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PHILOSOPHICAL TRANSACTIONS.

I. *The Influence of Stress and Strain on the Action of Physical Forces.*[*]

By HERBERT TOMLINSON, *B.A.*

Communicated by Professor W. GRYLLS ADAMS, *M.A.*, *F.R.S.*

Received April 5,—Read April 28, 1881

ORIGIN AND PURPOSE OF THE INVESTIGATION.

MORE than six years ago, whilst collecting together the results of the most trustworthy measurements of the various physical constants, with a view of establishing certain relationships which I conceived to exist between them, I was much struck with the discrepancies which exist, not only between the observations of different experimenters, but also frequently between those of the same individual. Many of these discrepancies, no doubt, arise from differences in the purity of the substances employed; but, when all due allowance has been made for such a cause, there still exists a large margin, which can only be accounted for by assuming that mere alteration of molecular aggregation must modify always, and in many cases considerably, the action of physical forces. This assumption has been already abundantly justified by the experimental researches of several eminent philosophers in every part of the domain of physical science; but, though much has been done, very much more remains to be done, and a wide field lies open before us which cannot fail to richly reward the patient explorer. I write patient explorer advisedly, because, in many cases, any difference of molecular disposition which we can effect with the means at our disposal, produces only a very minute alteration in that particular physical quality which we may be examining, and an alteration which we can only hope to measure accurately by the exercise of a large amount of perseverance and conscientious labour. But whether the effect to be observed be large or small, there is no doubt that further investigations of the kind here indicated must be made ere we can gain a true insight into the nature of the action of physical forces or into any relationships which may exist between them.

---

[*] In conducting these investigations I have been aided by a grant from the Government Research Fund of £4000. For this assistance, which has and will be of the greatest service to me, I here return my grateful thanks. I feel myself also under considerable obligations to Sir WILLIAM THOMSON, whose valuable paper on "The Electrodynamic Qualities of Metals" has really formed the ground-work of this memoir. The drawings have for the most part been carefully executed by Mr. J. E. JORDAN, of the Mining Record Office; and for the intelligent carrying out of the various details of the apparatus I am indebted to Mr. KIESER, of ELLIOTT Bros., and to Mr. FURSE, the Curator of the Physical Museum at King's College, London.

I proposed to myself, therefore, to examine as far as possible each of the various physical properties of one and the same specimen of different kinds of matter; and, further, to investigate the alterations which can be produced in these properties by stress and strain; being convinced that, by so doing, much light would be thrown on such subjects as electrical conduction, magnetic induction, thermal conduction and expansion, thermo-electricity, specific heat and elasticity. The words "stress" and "strain" are here used in their widest sense as intending to denote respectively the equilibrating application* of *any* physical force, and the definite alteration of form or dimensions experienced thereby by the matter acted on by the force.

In furtherance of the above-mentioned objects I procured some 60 feet of wire made of each of the various metals in common use, and also of the rarer metals, silver and platinum, and of the alloy platinum-silver, the three last in a state of chemical purity, and proceeded to determine one after the other the various physical constants of each.

In consequence of the extensive scope of these researches, several of the various parts into which the paper is divided are far from being as complete as I could wish, and hope at some future period to make them; but I venture to present them as they are to the Society, as tending to show still further than has hitherto been done the "correlation of the physical forces," and the value of this mode of investigation wherein one experiment immediately suggests several others.

PART I.—MODULI OF ELASTICITY.

"YOUNG'S MODULUS."

Description of apparatus.

The values of "YOUNG'S modulus" for the various metals were determined by a method devised by Sir W. THOMSON.† Wires of the same material and diameter are suspended in pairs about 1 inch apart from each other, and are attached by one extremity of each to the same support, the other extremities being fastened in the one case to a scale-pan, and in the other to the centre of a bar of wood or metal carrying constant equal weights at each end; the latter wire is provided with a scale, and the former with an index of some sort which, being level with and close to the scale, serves to measure any alteration of length produced by weights placed in the pan. By this simple and ingenious arrangement, any errors which might otherwise result from a slight yielding of the support, or from changes of temperature, are avoided.

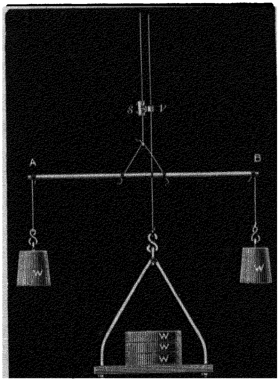
---

* 'Brit. Encyc.,' Art.: "Elasticity," p. 24.

† Proc Roy Soc, vol. xxix., p. 221.

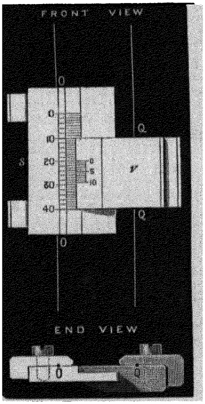
In my own particular experiments, the length of each wire between the support, which consisted of a vice firmly screwed into a stout wooden plank, and the scale and index was about 30 feet. To the extremities of the metal bar (fig. 1, A B), which was supported by one of the pair of wires, were attached two weights, each equal, in most cases, to one-fourth of the breaking-weight of the wire. To this wire was clamped by two screws a silvered metal scale (fig. 2, S), divided into half-millimetres. To the other wire was clamped in the same manner a vernier, V, reading to $\frac{1}{20}$th of a millimetre. This vernier was forked (fig. 3), so that, though capable of free up-and-down motion, it could not readily be dislodged sideways. By means of a compound microscope, an alteration of length of the wire equal $\frac{1}{100}$th of a millimetre could readily be estimated.

Fig. 1.



One-tenth size.

Figs. 2 and 3.



Scale and Vernier, full size.

Mode of experimenting.

In the case of the harder wires the one to be tested was subjected for several days to the stress produced by a weight three-fourths of the breaking-weight; the other wire, which may be called the comparison-wire, in the meantime sustaining a load equal one-half of the breaking-weight. During this time the relative positions of the scale and vernier were viewed from day to day; and when there seemed to be no alteration of these positions, the wire sustaining the heavier load was relieved from stress and allowed to rest for a period of from one to six days. Weights were then very carefully* put on to or removed from the pan attached to this wire, and the changes thereby produced in the length determined, the wire having in the meantime been permanently loaded to a sufficient extent to keep it quite straight. The softer metals—lead, tin, and even zinc and aluminium—were subjected to stresses much less in proportion to their tenacities than those used with the other metals. In their case each of the pair of wires was elongated permanently about 10 per cent. by weights, half the weight was then removed from the comparison-wire, and the two allowed to rest for four or five days. After this interval, the other wire was entirely relieved from stress, and finally was permanently loaded with a weight equal to that on the comparison-wire. As with the harder metals, the real testing did not begin till time wrought no appreciable change in the relative lengths of the two wires.

Reasons for the above precautions.

In the case of all the substances employed very great care was taken with the determinations, which often numbered more than one hundred for each of two or more different weights, and it was noticed after the experiments had been carried on for some time that the results obtained on different days frequently varied to a greater extent than could be accounted for by mere errors of observation. At first it was suspected that, in spite of the mode of suspension, a slight yielding of the upper support might be the cause of these discrepancies. Two additional wires were therefore suspended to this same support and loaded alternately with weights of 14 lbs. so as to tend to turn it about axes, respectively parallel and perpendicular to the line joining the points of attachment of the first two wires. Not the slighest effect, however, could be detected on the relative positions of the scale and vernier. The latter were then both shifted several time to parts of the wires a few inches above or below their previous positions to ascertain if any sticking between them could be observed—though in all cases care had been previously taken to adjust them so that their planes were parallel to that of the wires. But again these alterations did not seem to affect the results. The wires were now taken down and examined in case there might have been some slight slip of their ends through the vice. No sign, however, of anything of the kind could be detected, and on re-suspending and again testing them almost exactly the same results were obtained. Finally, the permanent loading was gradually increased in amount to determine if any error had arisen from the wires not having been sufficiently straight. It was hardly expected that this would prove to be the case, as before any of these trials had taken place the wires had been previously loaded for at least 24 hours to such an extent as to cause considerable permanent elongation. Nor indeed could any appreciable change in the measurements be detected until the weights used were such as to produce *further permanent elongation.*

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* Several methods were tried for loading and unloading; amongst other the stress was applied by allowing measured amounts of water to flow slowly into a large but comparatively light pail attached to the wire instead of the scale-pan, the removal of the stress being accomplished by suffering the water to pass out through a piece of caoutchouc tubing attached to an orifice at the bottom of the pail, and which during the process of loading was hitched up by the side of the pail. I found, however, that quite as good results could be obtained by putting on the weights by small amounts at a time by hand; a little practice, as far as my experience goes, enables the experimenter, especially if a support be provided for the elbow, to load or unload without causing any appreciable shock.

When this happened, however, there was evidently an apparent alteration of elasticity. A reference to the notes made on previous occasions then revealed the fact that *whenever the above-mentioned discrepancies had occurred a similar permanent elongation had been produced.* A fresh series of experiments was therefore begun, which ended in the discovery of the following facts :—

1. In the case of a wire which has suffered permanent extension the temporary elongation which can be produced by any load becomes less as the interval between the period of permanent extension and that of applying the load becomes greater.*

2. This increase of elasticity is greater in proportion for great loads than for small ones.

3. The increase of elasticity takes place whether the wire is allowed to remain loaded or unloaded between the period of permanent extension and that of the testing for the elasticity.

4. The *rate of increase* of the elasticity varies considerably with different metals; with some the maximum elasticity is apparently attained in a few minutes, and with others not till some days have elapsed—iron and steel are in this last respect very remarkable.

5. The elasticity can also be increased by heavily loading and unloading several times, the rate of increase diminishing with each loading and unloading.

6. A departure from "HOOKE's law" more or less decided always attends recent permanent extension, even when the loads employed to test the elasticity do not exceed one-tenth of the breaking-weight.†

7. This departure is diminished very noticeably in the case of iron, and much less so with the other metals by allowing the wire to rest for some time either loaded or unloaded; it is also diminished by repeated loading and unloading.

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* Since writing the above I have found that the gradual increase of elasticity with time in the case of soft-iron wires has been also noticed by EWING (Proc. Roy. Soc., 1880, No. 205, vol. xxx., p. 510.)

† G. WIEDEMANN and WERTHEIM have proved that there is a similar departure from "HOOKE's law" in the rigidity of metals when tested by the method of statical torsion. I shall have occasion in Part III. to refer further to Professor WIEDEMANN's valuable researches relative to torsion and magnetism: but see WIEDEMANN's 'Annalen,' 1879, vol. vi.; Phil. Mag., vol. ix., Jan. and Feb., 1880; 'La Lumière Electrique,' vol. vi., Nos. 2, 3, and 4; or WIEDEMANN's 'Galvanismus,'

The above statements apply in a greater or less degree to all the metals employed, but it will suffice perhaps to give in illustration some experiments with iron and copper wires. The pan employed in all these investigations weighed 2 kilogs., and, except in one or two instances which will be mentioned, was never detached from the wire. In the following experiments a + number will signify an increase, and a — number a decrease of length. In all cases the readings are given in half-millims., and the weight of the pan is not included in the estimate of the load.

The following series of experiments was made with an iron wire 0·62 millim. in diameter :—

Experiment I.

A weight of 12 kilogs. was put upon the pan and had the effect of permanently elongating the wire about 1½ per cent. The scale and vernier were now fixed to the wires and the following readings taken :—

No. of kilogs on pan	Reading of scale	Alteration of length.	Mean values
12	30·10		
0	5·70	−24·40	24·35[*]
12	30·50	+24·80	
0	6·20	−24·30	
8[†]	21·18		
0	5·08	−16·10	16·10
8	21·15	+16·07	
0	5·02	−16·13	
4	13·00	+ 7·98	7·95
0	5·08	− 7·92	
4	13·02	+ 7·94	

It appears therefore that the first 4 kilogs. would produce an alteration of 7·95, the second four of 8·15, and the third four of 8·25 divisions of the scale.

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[*] The + numbers are never taken in estimating the mean of the values of the alterations produced by the largest weight, in order to avoid the effect of permanent set.

[†] Scale slightly shifted by accident.

Experiment II.

The same wire having been elongated by traction to the extent of 29·4 centims. was again tried with the same weights immediately afterwards.

No of kilogs. on pan.	Reading of scale.	Alteration of length.	Mean values
8	21 10		
0	3·58	−17·52	17 52
8	21·10	+17 52	
0	3·58	−17 52	
4	12 08	+ 8·50	8 50
0	3 58	− 8 50	
12	31·10	+27 52	26 90
0	4 20	−26·90	
The scale-pan was now removed for 1 minute and then put on again			
12	30·10		
0	3 12	−26 98	26 98
8	20·50	+17 38	17·37
0	3 15	−17 35	
4	11 60	+ 8 45	8 45

Taking only the last trials we see that, now, the first 4 kilogs. produce an alteration of 8·45, the second of 8·92, and the third of 9·61.

Experiment III.

The wire was still further lengthened by 11·8 centims. and the pan being on the reading of the scale was 3·40. The pan was then removed for one minute, and afterwards replaced, scale now 2·40. Again the pan was removed for 30 minutes, and replaced, scale 2·18. The trials were then renewed with the same weights as those used in the other experiments.

No. of kilogs on pan.	Reading of scale.	Alteration of length	Mean values.*
0	2·18		
4	10 80	+ 8 62	8 48
0	2·32	− 8 48	
8	20 30	+17 98	17 65
0	2 65	−17·65	
12	30 90	+28 25	27 50
0	3·40	−27 50	
4	11·90	+ 8 50	8·50
8	21·00	+17·60	17 50
0	3·50	−17 50	
12	30·90	+27·40	27·30
0	3 60	−27 30	

Taking the results of the last observations we obtain for the first 4 kilogs. an alteration of 8·50, for the second four 9·00, and for the third 9·80.

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* In this and the following tables the − numbers only are taken, because it was always found that after the wire had been entirely relieved from stress a small sub-permanent set was produced by each of the weights when put on for the first time after such a release.

Experiment IV.

The wire having been left for two days without weights or scale-pan was again tested with the pan on.

No. of kilogs. on pan.	Reading of scale.	Alteration of length	Mean values.
0	3·30		
4	11·58	+ 8·28	8·23
0	3·35	− 8·23	
Scale-pan removed for 1 minute and then replaced.			
0	3·30		
8	19·80	+16·50	16·42
0	3·38	−16·42	
Pan again off and on.			
0	3·30		
12	28·10	+24·80	24·70
0	3·40	−24·70	
Pan off and on.			
0	3·30		

Here we see that the effect of the first 4 kilogs. is represented by 8·23, the second four by 8·19, and the third by 8·28.

As it had been found by preliminary experiment that the density of the wire was not permanently decreased to any extent which would introduce an appreciable error by supposing it to remain constant, we can easily make the different experiments comparable with each other by assuming that the permanent change of section is proportional to that of the length. If we do so we arrive at the following results:—

No of experiment	Temporary alteration of length produced by the load.	Load in kilogs.	Average alteration per 4 kilogs. on unit of area.*	Mean alteration per 4 kilogs in each experiment
I	7·95	4	7·95	8·04
	16·10	8	8·05	
	24·35	12	8·12	
II	8·16	4	8·16	8·41
	16·81	8	8·41	
	26·01	12	8·67	
III.	8·11	4	8·11	8·38
	16·70	8	8·35	
	26·05	12	8·68	
IV	7·85	4	7·85	7·84
	15·66	8	7·83	
	23·56	12	7·85	

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* The unit of area is supposed to be that of the section of the wire in I.; the length tested was the same in all the experiments.

Remarks on the preceding experiments.

It appears from the last table that the temporary elongation produced in an iron wire by a load of given magnitude becomes greater as the permanent elongation becomes greater up to a certain limit of the latter, which limit seems to depend upon the load used to produce the temporary effect. When the above-mentioned limit has been reached further permanent elongation begins to increase the elasticity, and this increase, as other experiments have shown, is continued up to the breaking point of the wire.

The increase of elasticity produced by rest, which is very conspicuous when we compare III. and IV., is the more remarkable as it is not attended, as was at first supposed would be the case, by any appreciable permanent shortening; the latter amounting in the present instance to only ·1 millim. out of a length of 8600, actually not ·002 per cent.

This phenomenon is moreover evidently closely allied with one noted by BOTTOMLEY,[*] who has recently discovered that in the case of iron the permanent elongation which can be produced by any weight may be very largely diminished by putting on this weight in small quantities at a time with intervals of rest between, and also that the breaking stress may be considerably increased by the same process.

We may assume that the mutual attraction existing between the molecules of a wire will always tend to make them take up such positions as will give a maximum mutual attractive force. When, therefore, a wire has been permanently stretched, the molecules would immediately take up these positions were it not for coercive force; this, however, causes delay, so that if the wire were tested shortly after the permanent extension has taken place, the elasticity would be found to be less than when the molecules have had sufficient time to finally settle themselves. Nor is it necessary that any appreciable permanent contraction of the wire should attend this increase of elasticity, as the mere change of arrangement of the molecules would suffice for the purpose. To a similar cause is no doubt also due the increase of portative power which can be produced in a permanent magnet by gradually increasing its load.

Another point to be noted is the sub-permanent set which is produced in all wires; this set is greater, according as the load permanently left on the wire is greater, and also increases up to a certain limit with the time during which the stress producing the set is applied: from this it follows that the readings taken on loading a wire step by step to a certain amount will be different from those taken at the same stages of stress on unloading. This fact has already, I believe, been noticed by THOMSON; but as I wish particularly to draw attention to it in the case of iron, I give the results of one out of several experiments made on iron and copper with a view of testing the matter.

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[*] Proc. Roy. Soc., vol xxix, p. 221.

Experiment V.

Annealed iron wire which had been very heavily loaded and unloaded a great many times on different days previous to this last experiment.

No. of kilogs. used for load	Scale reading on loading by 4 kilogs. at a time.	Scale reading on unloading by 4 kilogs. at a time.	Difference
0	21 22	21 22	·00
4	18 10	18 00	·10
8	15 20	15 01	·19
12	12 38	12 10	·28
16	9 45	9 22	·23
20	6 50	6 50	00

This last experiment shows very clearly that the wire does not recover itself until all the load has been removed. Also in Experiment IV. we see that a certain amount of set disappears even with the removal of the comparatively small load of the scale-pan.

With most of the other metals the recovery of elasticity is much less marked after the first hour than is the case with iron.

The following experiments were made with a soft copper wire ·81 millim. in diameter and 630 centims. in length:—

Experiment VI.

The wire was loaded for a few minutes with a weight slightly over 8 kilogs., and, on the removal of stress, the following observations were made:—

No. of kilogs.	Time after permanent elongation.	Reading of scale.	Alteration of length.
8	4 minutes	5 80	
0	6 "	19 45	−13 65
8	8 "	5 95	+13 50
0	10 "	19 10	−13·15
8	12 "	5 80	+13 30
0	14 "	18 80	−13 00
8	16 "	5 75	+13·05
0	18 "	18 70	−12 95
8	20 "	5 68	+13 02
0	22 "	18 60	−12 92
8	17 hours	5 65	
0	17 hours 2 minutes	18 50	−12·85
8	" 4 "	5 60	+12 90

A great part of the gradual increase of elasticity here observed is due to loading and unloading, and a similar effect may be observed in Experiments II. and III., but part is evidently due to the influence of mere rest; this is best shown by taking the differences between consecutive + or − alterations for the different times.

It will also be observed that the + and − values both here and in the experiments on iron gradually become equal under the influence of rest and loading and unloading.

Cases of aluminium and zinc.

With these metals[*] both the maximum temporary increase of length, caused by putting on weight, and the recovery on the removal of stress, are attained slowly in comparison with most metals. An illustration of this is afforded in the next experiment, which was made on an aluminium wire.

Experiment VII.

No. of kilogs.†	Reading of scale	No of minutes after putting on or taking off load	Alteration of length in half millims.
0	1·4		
6	18·9	1·7	+17·5
	19·2	3·0	+17·8
	19·5	5·0	+18·1
	19·8	8·5	+18·4
	20·1	12·0	+18·7
	20·4	18·0	+19·0
	20·7	26·3	+19·3
	20·8	30·0	+19·4
0	4·4	5	−16·4
	3·7	1·5	−17·1
	3·4	6·0	−17·4
	3·0	25·0	−17·8
	1·9	1440·0	−18·9
6	19·0	1·0	+17·1
	19·3	3·0	+17·4
	19·6	5·0	+17·7
	19·9	10·5	+18·0
	20·2	18·0	+18·3
	20·5	26·0	+18·6
	20·8	40·0	+18·9
	21·1	52·0	+19·2
	21·15	60·0	+19·25
0	4·35	1·0	−16·80
	3·00	30·0	−18·15
	2·20	1440·0	−18·95

It must be observed that during the whole of this time the comparison-wire was loaded with a permanent weight equal to that on the wire which was being tested, and that the stress thus produced had been acting for several days previous to these trials.

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[*] Probably tin and lead if they had been loaded sufficiently would have also behaved in this manner, it was impossible, however, to use any but very light weights in determining their elasticity, as otherwise permanent set would always have been produced.

[†] A weight of 4 kilogs. was kept permanently on the scale-pan.

When lesser weights were now used and only the scale-pan left on permanently the maximum alterations took place much more quickly, and the departure from "HOOKE'S law," which had before been very considerable, became comparatively slight.

The modulus was calculated from these last results.

Cases of tin and lead.

These metals are remarkable for the manner in which they run down under the influence of the slightest stress, and also for the persistence of this running down; in this latter respect they surpass aluminium and zinc. It was necessary, therefore, to keep them loaded for many days before attempting to determine their elasticity. The weights also employed for this latter purpose were very small, and, in consequence, the values of the modulus obtained by using them cannot be considered as accurate as those of the other metals; I believe, however, that they are correct within 2 per cent.

"HOOKE'S LAW."

We have seen that there is a very appreciable departure from this law shortly after permanent extension has taken place; but a departure also exists when a long rest has been allowed, both when the wire has in the meantime been heavily weighted and when it has not, even though only moderate loads be employed.

Thus, in the case of a soft copper wire capable of bearing a load of 18 kilogs., and which had been heavily loaded and frequently tested during a period of three weeks, the following observations were made:—

Experiment VIII.

Number of kilogs. in load.	Average alteration per kilog.
	Millimetre.
2	·808
4	·811
6	·816

The values here recorded are the means of about 20 observations on each weight made during the last three days of observation, and show small but decided differences.

Again, another piece of copper similar to the above was treated in the same manner, but for a period of six days, and when examined on the last day, with loads up to 12 kilogs., gave the following results as the means of seven trials with each weight:—

Experiment IX.

Number of kilogs. in load	Average alteration per kilog.
	Millimetre.
2	·830
4	·835
6	·840
8	·847
10	·866
12	·911

It is here seen that the alteration per kilogramme rapidly increases when the loads become heavy, and yet in both these last experiments the wire recovered its original length on the removal of the load. It is evident, therefore, that in the case of annealed copper the length increases in greater proportion than the load, and this was proved to be the case whether the wire was allowed to rest loaded or unloaded.

Similar results were obtained with annealed platinum, silver, aluminium, platinum-silver, German-silver, and zinc. With soft iron, however, the case is different if the wire be weighted for some time after permanent extension has taken place. An examination of Experiment XIII. shows that under these circumstances the average alteration decreases up to a certain point as the load increases.

It remains now to consider the case of iron allowed to rest unloaded.

Experiment X.

An annealed iron wire, after having been permanently elongated by traction about 8 per cent., remained unloaded for several days, and was afterwards tested with weights up to 10 kilogs. :—

Number of kilogs in load.	Alteration of length per kilog.
	Millimetres.
1	1 350
2	1 355
3	1·367
4	1·376
5	1·386
6	1·388
7	1·386
8	1·389
9	1·392
10	1·394

Here the average alterations of length increase with the load, though not to the same extent as with copper.

In both the last experiments with iron the recovery of the wire after the removal of the stress was so perfect that the zero position of the vernier was not shifted one-tenth of a millimetre.

DISCUSSION OF WERTHEIM'S EXPERIMENTS ON ELASTICITY.*

The values of "YOUNG'S modulus" obtained by WERTHEIM by vibrations, longitudinal or transverse, are generally larger than those got by static extension; and these differences are considerably greater than those which would be produced by the heating and cooling effects of contraction and elongation.

Sir W. THOMSON says† that "it is probable that his (WERTHEIM'S) moduli, determined by static elongation, are minutely accurate; the discrepancies of those found by vibrations are probably due to imperfections of the arrangements for carrying out the vibrational method." I venture, however, to believe that the main cause of the above-mentioned discrepancies is to be found in WERTHEIM'S *mode of proceeding* when determining the elasticity by *static extension.* The plan adopted by him was to put on a weight, take a reading with the measuring microscope, and, after removing the weight, take a second reading, the difference between these two readings being used in determining a value for the modulus. The same operations were repeated with greater and greater loads until the wire underwent very considerable permanent extension, and was in many cases broken. The mean of all the values thus obtained was taken to represent the true one.

Now, if, after considerable extension had taken place, WERTHEIM had repeated his trials with each of the previous weights, he would have obtained appreciably different values, and the general result would have been to give him a greater mean value for the elasticity. Moreover, my experiments have shown, as we have seen, that, even if all precautions be taken, different loads will give different values for the elasticity.

The best way of comparing the methods of static extension and longitudinal vibrations would be to determine, first, the elasticity by the former method with small loads, and then to use the latter method with the same wire under as nearly as possible the same conditions of tension. I hope at some future time to be able to make further experiments in this direction; but in some few trials with copper, iron, steel, and German-silver I have obtained values for the elasticities by the two methods which accord more nearly with each other than those got by WERTHEIM.

In order to ascertain whether the influence of rest—which in iron is so marked in increasing the value of "YOUNG'S modulus" as determined from static extension—would be equally or at all apparent when longitudinal vibrations are employed, several experiments were made on iron by the latter method, both the syren and APPUNN'S tonometer being employed for the purpose of counting the number of vibrations. Both these instruments gave very consistent results, and could be depended upon within at least ½ per cent.; yet no difference could be detected between the note of the wire after recent permanent extension and that after a rest of 24 hours. As it was thought that perhaps the act of vibrating the wire might immediately produce the same effect as continued rest, a fresh pair of iron wires were suspended and tested in the usual manner with the scale and vernier; but it was ascertained that vibrating a wire under these circumstances did not produce any immediate appreciable effect on the elasticity. It would appear, therefore, that the effect of rest is not felt when the temporary elongations are very small.

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* Ann. de Chim. et Phys., tom. xii., 1844.

† 'Brit. Encyc.,' Art.: "Elasticity," § 77.

Permanent Alteration of Elasticity Produced by Permanent Extension.

Experiment XI.

The same wire as in Experiment X. was further lengthened by 7, 15, 10 and 7½ centims. respectively on four separate occasions, and after each elongation the load was removed, and a rest of 24 hours allowed. After each rest the temporary alteration of length produced by 8 kilogs. was determined, the vernier after each permanent extension having been shifted to its original position.

Actual alteration observed.	Calculated alteration which would be produced on wires of the same section as in Experiment V.	Total percentage of permanent extension produced before testing.
12·88	12·88	4·0
13·02	12·87	5·1
13·32	12·87	7·5
13·50	12·84	9·0
13·65	12·83	10·2

These results are the means of several observations in each case, and show that the elasticity of copper is very slightly increased by these particular amounts of permanent extension, when the wire is allowed to rest unloaded.

A similar effect was proved to be produced on copper wire which was kept loaded for some time after permanent extension.

Experiment XII.

A piece of annealed iron wire, 860 centims. in length, was elongated by traction to the extent of 21·7 centims. so as to make it perfectly straight, and afterwards allowed to rest unloaded for two days; a set of experiments was then made which resulted in giving a mean value of 8·28 half-millims. as the alteration produced by the first 4 kilogs., 8·23 for the second, and 8·29 for the third, and an average on the whole of 8·27 for 4 kilogs.

The wire was now further lengthened by 13·3 centims., and the vernier shifted so that the same length of wire as before was under examination, and again a rest of two days allowed. On loading the wire with the same weights as before, an alteration of 8·50 was produced by the first 4 kilogs., 8·48 by the second, and 8·50 by the third, giving an average alteration of 8·49 for 4 kilogs. Allowing for the permanent diminution of section, the last alteration would correspond to 8·36 for 4 kilogs. on a wire of the same section as that previous to the second permanent elongation.

We have, in the case of this wire, therefore, a small but decided *diminution* of elasticity produced by *this amount* of permanent extension, when the wire is allowed to rest *unloaded* after the extension has taken place.

Experiment XIII.

An annealed iron wire of the same length as the previous one was tested in the same manner, except that after each permanent extension it was loaded with a weight of 20 kilogs., and this load was suffered to remain on the wire for 24 hours.

Total percentage of permanent extension ..	1·0	4·9	8·0	8·9*
Load in kilogs	Average alteration of length in half-millims per load of 2 kilogs. on unit area †			
2	1·590	1·571	1·554	1·522
4	1·579	1·571	1·514	1·516
6	1·557	1·557	1·510	1·494
8	1·544	1·545	1·504	1·497
10	1·540	1·537	1·502	1·484
12	1·539	1·529	1·495	1·482
14	1·543	1·525	1·486	1·482
16		1·515	1·490	1·477
18			1·492	1·477
20				1·476

It appears, therefore, that in the case of annealed iron the elasticity is *increased* by permanent extension if the wire be allowed to remain heavily loaded for some time after such extension has taken place.

Moreover, it is remarkable that when the wire has been treated in the above-mentioned manner the average alteration per unit load *diminishes*‡ as the load employed for testing *increases*, whereas with all the other annealed metals similarly treated, exactly the opposite effect is produced. This peculiarity of iron is no doubt to be attributed to its superior coercive force; and to the same cause must probably be assigned the difference between the effect of permanent extension on the elasticities of iron and copper when these metals are allowed to rest unloaded after the extension has taken place.

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* The wire was broken at a point about 3 inches from the scale-pan by this last extension.

† The unit area is assumed to be the area of the section of the wire after the last permanent extension; this area was 00137 square centim.

‡ [Note added April, 1882.—It should be stated here that a load of about 6 kilogs. (not included in the loads given) was left permanently on the wire. I have since found by an indirect method (see Part II.) that, if the wire be entirely relieved from stress before beginning to test for the temporary effect of loading, the temporary elongation increases with the first few loads in greater proportion than the latter. We may say, therefore, that in the case of iron wire which has suffered very considerable permanent extension and afterwards been allowed to rest for some time either unloaded or loaded, the ratio of the temporary elongation to the load producing it first increases with the latter to a certain limit, then diminishes to a second limit, and finally begins to increase again. If, however, the wire has during the interval of rest been heavily loaded, the first limit is reached more quickly than is the case when the wire has rested unloaded; so that if, as in this instance, it is necessary to leave even a comparatively small

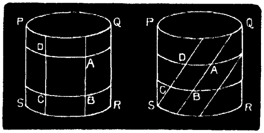
In order to examine still further the effect of leaving a heavy weight on the wire for long periods, 20 kilogs. were, after the above experiments had been made, left on the wire for one day, and then for two more days. The average increases per 2 kilogs. after each of these periods, when tested with 20 kilogs., were respectively 1·474 and 1·475 half-millims. Therefore the full effect of the loading must have been produced during the five days on which the previous trials had been made.

Finally, it should be observed that during the whole of these last experiments the wire returned to its original length on the removal of the load.

EFFECT OF PERMANENT TORSION COMBINED WITH TRACTION.

The above are the only *direct* experiments which were made of the effect of permanent extension on the value of "YOUNG's modulus;" but having ascertained indirectly that, at any rate in the case of some metals, permanent extension will, according to its amount, produce either decrease or increase of elasticity, I was induced to make a set of observations in which torsion was combined with longitudinal traction.

Fig. 4.



Let P Q R S, fig. 4,* represent a portion of the wire in the unstrained condition; and suppose that, the upper end having been fixed, the lower end is twisted in the contrary direction to the hands of a watch, thus causing the portion A B C D to be extended along the diagonal A C and compressed along the diagonal B D; if now a load be applied at the lower end S R, this will cause the wire to twist still further or to untwist, according as the extension produced by the load along A C is greater or less than that along B D.

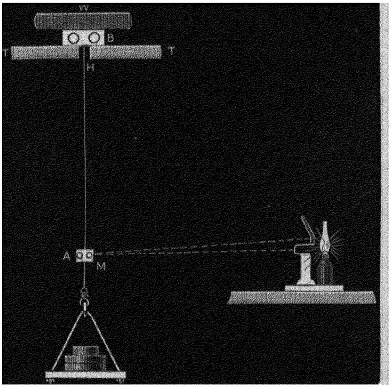
weight permanently on the wire, this weight may exceed that required for the above-mentioned limit. It is quite possible, also, that if the permanent extension and the heavy loading during rest be carried to very great excess, any load however small may exceed the first limit.]

* 'Electricity and Magnetism,' CLERK MAXWELL, vol. ii., p. 86.

Mode of experimenting.

The wire to be examined passed through a small hole, H, fig. 5, made in a stout table, and was clamped at its upper extremity into a brass block, B, the latter resting on the table and being sufficiently secured by a heavy weight, W, placed on the top of it. Near the lower extremity, which was looped to receive a scale-pan, was clamped a second small block, A, to one end of which was attached a light mirror of the kind employed with reflecting galvanometers; this mirror, used in connexion with a scale and lamp, enabled the observer to detect very small differences of torsion; the distance of the scale from the mirror was 1200 scale-divisions, and the length of each wire at the commencement of each experiment was 95 centims.

Fig. 5.



In the following table will be found the results of the experiments made with the different metals, these latter being for the most part pieces of the same wires as had been used in the determinations of "YOUNG's modulus," but in each case carefully re-annealed.

TABLE I.—Number of complete turns of permanent torsion produced before the application of the load $=n$. Temporary alteration of torsion produced by the load in terms of divisions of the scale $=a$; + signifies further twist, — untwist on loading.

a.

Iron, diameter = ·082 centim.

n.	*k.*	*a.*		
		5 mins. 0	½ hr. 0	16 hrs 0
10	2	5·0+	5·0+	6·3+
	4	9·0+	11·5+	12·3+
	6	8·5+	13·0+	19·3+
	8	8·0+	14·0+	22·5+
	10	3·0−	14·0+	..
		16 hrs. 0	16 hrs. 12	Average pan 2 kilogs.
50	2	12·5+	12·8+	12·8+
	4	..	20 3+	10 2+
	6	.	29 0+	14 5+
	8		37·2+	9 3+
	10	.	44·2+	8·8+
		16 hrs. 0		Average pan 2 kilogs.
130	2	23·0+	.	23·0+
	4	44 5+		22 3+
	6	61·0+	.	20·3+
	8	89·5+		22·4+
	10	95·0+	.	19·0+
	5 mins. 0			
280	2	20·7+		
	4	29·5+		
	6	32·0+		
	8	36·0+		.
	10	36·2+		

b.

Platinum, diameter = ·076 centim.

n.	*k.*	*a.*	
		5 mins 0	Average pan 2 kilogs.
50	2	75·5+	75·5+
	4	160·0+	80·0+
	6	251 5+	83·5+
		36 hrs. 2	Average pan 2 kilogs.
	2	74·0+	74·0+
	4	149 0+	74 5+
	6	237 0+	79 0+
		5 mins. 0	
100	6	322·3+	
		2 hrs. 6	Average pan 2 kilogs.
	2	103 5+	103·5+
	4	210·5+	105·3+
	6	327·0+	109 0+
		5 mins. 0	
150	6	412·5+	
		5 mins. 0	16 hrs. 0
250	6	477·0+	472·0+
		5 mins. 0	
300	6	514+	..
		5 mins. 0	
348	6	522+	..
	Broke.		

TABLE I. (continued)—Number of complete turns of permanent torsion produced before the application of the load $=n$. Temporary alteration of torsion produced by the load in terms of divisions of the scale $=\alpha$; $+$ signifies further twist, $-$ untwist on loading.

c

Copper, diameter = ·152 centim.

n	*k.*	*a.*		
		5 mins 0		
50	12	5+		
		5 mins. 0	¼ hr. 12	7 hrs. 12
150	2			7·5—
	4			15·5—
	6			25 5—
	8			
	10			43 5—
	12	42 5—	51·3—	53 5—
		5 mins. 0	1 hr. 12	14 days. 0
250	4		41—	49 5—
	8		87—	92 2—
	12	135—	139—	143 0—
		5 mins. 0	2 hrs. 12	Average pan 2 kilogs.
450	4		103·5—	51 8—
	8		200 5—	50 1—
	12	277—	304 0—	50 7—
490	Broke			

d.

Silver, diameter = ·076 centim

n	*k.*	*a.*
		16 hrs. 4
20	2	8+
	4	15+
120	2	83—
	4	170—
220	4	297—
320	2	285—
	4	547—
417	2	324—
	Broke	

e.

Aluminium, diameter = ·096 centim.

n	*k*	*a*
		2 hrs. 2
5	2	42+
20	2	142—
120	2	513—

f.

Tin, diameter = ·098 centim

n.	*k.*	*a.*
		5 mins. 0
5	·05	20—
10	·05	20—
20	·05	32—
40	·05	30—
80	·05	20—

g

Lead, diameter = ·098 centim.

n	*k*	*a*
		5 mins. 0
5	·05	8+
15	·05	5+
35	05	6+
80	05	0
	Broke	

h

Zinc, diameter = ·098 centim.

n.	*k.*	*a.*
		5 mins. 0
20	2	110+
60	Broke	

Table I. (continued)—Number of complete turns of permanent torsion produc before the application of the load $=n$. Temporary alteration of torsion produc by the load in terms of divisions of the scale $=\alpha$; $+$ signifies further twis $-$ untwist on loading.

k. Hard piano steel, diameter = ·082 centim			*l.* Copper, diameter = ·096 centim.		
n.	*k.*	*a.*	*n.*	*l.*	*a.*
		5 mins 0			5 mins 0
5	4	171−	10	6	103+
	8	193−	20	6	153+
	12	198−	40	6	151+
		12 hrs. 12			2 hrs. 9
5	4	162−	40	6	112+
	8	192−			
	12	192−			2 hrs. 12
Same wire partially annealed			40	6	53+
		2 hrs 12		10	100+
10	4	44·5−			½ hr. 13
	8	52 0−			
	12	47·0−	40	6	50 5+
				10	95 0+
Same wire completely annealed					½ hr. 14
		5 mins. 0	40	6	44 5+
15	4	29+		10	88 0+
	8	100+	Total permanent increase of length produced by the loading = 10 per cent		
		½ hr. 8			
15	4	34+			
	8	58+			

TABLE I. (continued)—Number of complete turns of permanent torsion produced before the application of the load $=n$. Temporary alteration of torsion produced by the load in terms of divisions of the scale $=\alpha$; $+$ signifies further twist, $-$ untwist on loading.

m			*n.*			*o.*		
Iron, a piece of the same hank as that used in *a*, but heated to a bright red and suddenly cooled			Copper, a piece of the same wire as used in *l*.			Iron, a piece of the same wire as that used in *a*, but elongated 10 per cent.		
n.	*k.*	*a.*	*n.*	*k.*	*a.*	*n*	*k*	*a*
		5 mins 0			5 mins. 0			5 mins 0
20	10	76+	140	6	103·5+	10	6	12·2+
120	10	160+	240	6	50·0+			
10 turns of torsion in the other direction.			340	6	38·0−			24 hrs. 10
					½ hr 11	10	6	17·0+
		5 mins. 0	340	6	102·5−			3 days. 0
	2	16+	Heated to redness by passing a burner several times up and down.			30	6	18·0+
	10	62+						
								12 hrs. 0
			wire cool.		5 mins. 0			
						70	6	7·5+
			340	6	50·0−		10	·8+
			390	6	55·5−			
								2 days 12
						90	8	7·0−
							12	7·0−

Explanation of and remarks on Table I.

The times given in the various columns represent approximately the intervals between the imparting of the permanent torsion and the testing with the loads, whilst the numbers below the times are the number of kilogs. with which the wire was weighted during these intervals; thus $\overset{\text{5 mins}}{0}$ means that the wire remained unloaded for five minutes after the permanent torsion had been applied, and $16\,{}^{\text{hrs.}}_{12}$ that the wire was tested 16 hours after the torsion, having in the meantime sustained a load of 12 kilogs. By unloaded we must understand that the scale-pan weighing 2 kilogs. remained on the wire, except in the cases of lead and tin, when a weight of ·05 kilog. was substituted for that of the pan.

We observe that the annealed metals may be divided into two classes; that iron, platinum, lead, and zinc, after suffering permanent torsion, *twist* temporarily[*] on loading, showing that the load produces greater temporary lengthening in the direction in which the torsion had produced extension than it does in the direction of compression. It seems, however, that if the permanent extensions previous to the loading be carried to excess, the wire will begin to *untwist* on loading; thus in *a* we observed that after 260 turns of permanent torsion the twist produced by loading has begun to diminish and in *o* that, when the wire had been considerably stretched before torsion, after 70 turns the twist is changed to untwist. Moreover, the results in *k* on hard and soft steel, combined with those in *o*, clearly show us that whether the distance between the particles be increased by mechanical means or by hardening by the process of heating and suddenly cooling[†] we ultimately arrive at a point where the twist on loading is changed to untwist.

The second class includes the metals, copper, silver, aluminium and tin; these, like the metals of the first class, at first *twist* temporarily on loading, but after a comparatively small amount of permanent torsion has been applied begin to *untwist* and continue to do so, as far as could be ascertained, until the wires will not bear any further twisting without instantly breaking; indeed, the copper in *c* was broken several times, and became so brittle and hard as to snap, like steel that has been heated to a white heat and suddenly cooled, and yet the wire apparently showed no decrease in the amount of untwisting on loading.

It will further be noticed that the average twist or untwist per kilogramme is nearly the same for the different weights employed, but other experiments on metals of the *first class* showed that when the load became excessive the average twist became less and less, and was finally converted into untwist when the load approached the breaking stress; and moreover whereas with smaller loads the *permanent* effect was in the case of both classes to cause *untwist*, with these larger stresses the wire commenced to *twist permanently:* this latter point is most easily proved with lead or tin, as comparatively small weights are required to break the wires made of these metals.

It may, I think, be fairly concluded from these and the previous experiments, that with all metals the longitudinal elasticity is *diminished* by permanent extension *carried to a certain point;* but *beyond this point increased.*

Analogous results have been obtained by THOMSON with respect to the torsional rigidity of metals[‡] and as we shall see latter on, the action of all physical forces is apparently affected in a similar manner by stress and strain whether these latter be due to mechanical force, to magnetization, or to change of temperature.

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[*] It is perhaps as well to observe here that only the variations of torsion produced by *unloading* are recorded in the table.

[†] Sir W. THOMSON has already ('Brit. Encyc,' Art.: "Elasticity," § 81) proved that hard steel wire *untwists* on loading, after suffering permanent torsion; but we see that if the steel be softened it acts like iron and *twists* on loading.

[‡] Proc. Roy. Soc., vol. xiv., p. 289, and 'Brit. Encyc,' Art.: "Elasticity," § 78.

The influence of rest on the wire, whether the metal is left loaded or unloaded, is also very noticeable in the case of iron, and less but sufficiently so with the other metals; thus we see in *a* that in the case of iron after 10 turns of permanent torsion the amount of twist for 8 kilogs. is five minutes after the permanent torsion, 8 divisions, 14 divisions after a rest of half-an-hour, and 22·5 divisions after a rest of 16 hours.

TORSIONAL RIGIDITY.

The torsional rigidity of the wires was determined by the method of vibrations. The vibrators were similar to those employed by Sir W. THOMSON in his experiments on the rigidity and viscosity of metals,[*] namely, thin cylinders of sheet brass, supported by a thin, flat rectangular bar. The wire to be tested passed perpendicularly through a hole in the middle of the bar, and was there soldered. The other end of the wire was soldered into a stout iron bar, firmly held in a vice attached to a rigid support.

EFFECTS OF PERMANENT TORSION AND ELONGATION ON THE MODULUS OF RIGIDITY.

THOMSON has proved[†] that the rigidity of a wire is diminished both by permanent longitudinal extension and by permanent twist. As it seemed desirable to ascertain whether rest would restore any of the rigidity thus lost, and also whether the influences of permanent extension and torsion would be greater on vibrations through large arcs than through small ones, a series of experiments was begun of which the following are examples.

Experiment XIV.

An iron wire was considerably stretched, and the times of vibration ascertained to be—

Number of minutes after stretching.[‡]	Time of vibration.
	Seconds.
5	3·040
25	3 025
35	3·019
45	3·015
1440	3·000
2880	2·996

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[*] Proc. Roy. Soc., vol. xiv., p. 289.

[†] 'Brit. Encyc.,' Art.: "Elasticity," §§ 78, 81.

[‡] The vibrations were generally counted in each trial for about 10 minutes, and the times are reckoned from the commencement of each trial. The initial arc of vibration was in each case 10°.

Experiment XV.

A copper wire was elongated by 10 per cent., and when vibrated gave the following results :—

Number of minutes after stretching	Time of vibration.
	Seconds.
1	4·348
11	4·337
120	4·330
1440	4·316

Experiment XVI.

An iron wire about 4½ feet long received 50 turns of permanent twist and was then tested :—

Number of minutes after torsion.	Time of vibration.
	Seconds.
1	3·0415
180	3·0200
1440	3·0000

Experiment XVII.

A copper wire about 4½ feet long received 50 turns of permanent twist and gave the following results :—

Number of minutes after torsion.	Time of vibration
	Seconds.
1	4·615
2	4·500
5	4·488
1440	4·444

Experiment XVIII.

The time of vibration of a copper wire which had suffered no permanent torsion was, when vibrated through a small arc, 6·242 seconds, and when started with a twist of three revolutions* 6·316 seconds, there being thus a difference of ·074 second. The wire was now subjected to 100 turns of permanent torsion, and in a few minutes afterwards vibrated once in 6·431 seconds through small arcs, and in 6·58 seconds when started with a twist of three revolutions. The difference between the two times is now ·149 second or twice the former difference. After a rest of 24 hours the two times became respectively 6·370 and 6·474, and the difference ·104 second.

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* Only 19 vibrations were counted in each trial, so that the amplitude of vibration might not be diminished too much.

Experiment XIX.

An iron wire, 4½ feet in length, was vibrated through different arcs for 30 seconds in each of several trials, and the means of these taken as the time of vibration.

Number of degrees in initial arc of vibration.	Mean time of oscillation.
°	Seconds = t.
10	1·200
90	1·250
180	1·244
90	1·228
10	1·200
360	1·210
180	1·188
720	1·250
360	1·245
180	1·220
90	1·182
10	1·200

Previous to these trials the wire had received 10 turns of permanent torsion, which had had the effect of diminishing the rigidity.

Vibrating the wire caused, in this instance, as it does in all cases where the metal has received permanent torsion, a certain amount of untwisting depending upon the amplitude of the arc of vibration.*

Experiment XX.

A piano-steel wire was vibrated several times, for about 30 seconds each time, through 1080°, and the time of vibration was found to diminish on each trial until the fourth, when it became constant. It was then vibrated through smaller and smaller arcs with the following results :—

Initial arc of vibration.	Time of vibration.
°	Seconds = t.
1080†	1·647
720	1·662
360	1·677
10	1·706

---

* In consequence of this untwisting the vibrations were, in experiments of this kind, counted from the *beginning of the swing*, and not, as is usual, from the position of equilibrium.

† The elasticity of the wire was perfect for this degree of torsion.

Remarks on the above experiments.

It appears from Experiments XIV.–XVIII. inclusive (1) that the loss of rigidity produced by twisting or stretching a wire beyond the limits of elasticity, is partly diminished by rest; (2) that the loss is more sensible with large arcs of vibration than with small ones; and (3) that the influence of rest is more apparent in the case of large vibrations than in that of small ones.

Experiment XIX. shows that continual vibrating through large arcs has a similar effect on the rigidity to that produced on the longitudinal elasticity by heavily loading and unloading, the time of vibration through large arcs being by the former process made less, just as the temporary elongations caused by heavy loads are diminished by the latter.

Finally, Experiment XX. shows that in the case of a wire possessing great coercive force, the effect of vibrating through a large arc for several minutes actually makes temporarily the rigidity, as determined from such vibrations, *greater* than that determined from smaller vibrations: an effect analogous to that produced by leaving a wire heavily weighted for some time, when, as we have seen, the temporary effect on the length of large loads is less in proportion than of small ones.

We thus see that the effect of permanent torsion on the torsional rigidity is similar in every respect to the effect of longitudinal extension on the value of YOUNG's modulus."

Iron, aluminium, copper, and silver are the only metals which have, as yet, been tested in the above-mentioned manner, and iron, as before, is conspicuous for the large influence on it of continued rest.

In Table II. will be found embodied the results obtained in the case of each substance for the modulus of rigidity and "YOUNG's modulus," together with some other data which are further supplemented in Table III.*

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* For observations on the moduli of elasticity of nickel and carbon at the ordinary temperature of the room, and of iron, steel, nickel, and copper at the temperature at 100° C., see Part II.

TABLE II.

Name of metal.	Condition.	Specific gravity of water at 4° C. = 1. Δ.	Torsional rigidity in grms. per square centim. r.	YOUNG's modulus in grms. per square centim. e.	Ratio of lateral contraction to linear elongation. σ.
Iron (1)*	Annealed . .	7·759	$773\cdot1\times10^{6}$	1981×10^{6}	·281
Iron (2)	Hard drawn .	7·740	$771\cdot1\times10^{6}$	2041×10^{6}	·325
Iron (3)	Hard drawn .	7·520	$637\cdot2\times10^{6}$	1683×10^{6}	·321
Piano steel (1) . . .	Hard drawn .	7·814	$746\cdot5\times10^{6}$	1894×10^{6}	·269
Piano steel (2) . . .	Hard drawn .	7·784	$782\cdot3\times10^{6}$	1968×10^{6}	·259
Platinum (1)	Hard drawn .	21·323	$686\cdot4\times10^{6}$	1443×10^{6}	·051
Platinum (1)	Annealed . .	21·300	$692\cdot7\times10^{6}$	1490×10^{6}	·076
German-silver (1) . .	Annealed . .	8·700	$493\cdot7\times10^{6}$	1335×10^{6}	·354
German-silver (2) . .	Annealed . .	8·632	$456\cdot2\times10^{6}$	1291×10^{6}	·415
German-silver (2) . .	Hard drawn .	8·632	$389\cdot6\times10^{6}$	1169×10^{6}	·500
Copper (1)	Annealed . .	8·913	$440\cdot6\times10^{6}$	1160×10^{6}	·315
Copper (1)	Hard drawn .	8·896	$418\cdot2\times10^{6}$	1449×10^{6}	·733
Copper (2) . . .	Annealed . .	8·851	$419\cdot3\times10^{6}$	1218×10^{6}	·453
Copper (3) . . .	Annealed . .	8·825	$457\cdot4\times10^{6}$	1143×10^{6}	·293
Platinum-silver (1)† .	Annealed . .	12·623	$369\cdot9\times10^{6}$	1051×10^{6}	·420
Platinum-silver (1) .	Hard drawn .	12·608	$302\cdot3\times10^{6}$	1038×10^{6}	·717
Brass (1)	Hard drawn .	8·396	$321\cdot1\times10^{6}$	$988\cdot4\times10^{6}$	·587
Brass (2)	Hard drawn .	8·488	$332\cdot5\times10^{6}$	$988\cdot1\times10^{6}$	·504
Zinc (1)	Hard drawn	7·138	$338\cdot4\times10^{6}$	$766\cdot9\times10^{6}$	·133
Silver (1)	Annealed . .	10·491	$271\cdot8\times10^{6}$	$742\cdot4\times10^{6}$	·367
Silver (1)	Hard drawn	10·434	$274\cdot6\times10^{6}$	$764\cdot5\times10^{6}$	·392
Aluminium (1) . .	Hard drawn .	2·730	$249\cdot8\times10^{6}$	$669\cdot4\times10^{6}$	·340
Aluminium (1) . . .	Annealed .	2·732	$265\cdot2\times10^{6}$	$673\cdot1\times10^{6}$	·269
Tin (1)	Drawn . .	7·264	$120\cdot9\times10^{6}$	$277\cdot1\times10^{6}$	·145
Lead (1)	Drawn . . .	11·193	$74\cdot0\times10^{6}$	$167\cdot0\times10^{6}$	·136

Remarks on Table II.

The determinations of r were made in all cases with unstretched pieces of the different wires, and may for the most part be considered as extremely accurate, but in the cases of tin and lead it was found very difficult to obtain good observations on account of the great viscosity of these metals; indeed, with the former only *four* vibrations of convenient amplitude could be counted.

Tin, lead, zinc, and aluminium are placed in the order of their viscosity.

The annealed and hard drawn wires having the same numbers attached to them in the tables are not the *same pieces* but are cut from the *same hank.* I should have employed actually the same pieces in the two conditions, had I not wanted them for the purposes mentioned in Part II.

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* This metal and copper (3) I obtained through the kindness of Sir W. THOMSON; their moduli of elasticity had been carefully determined by T. GRAY, in the Physical Laboratory of Glasgow University. Iron (2), (3), steel (1), (2), and brass (2) were tested some years ago by myself with the cathetometer.

† This alloy was composed of two parts by weight of silver and one of platinum.

The values of Δ were determined very carefully, more so perhaps than was necessary. The specimens used for this purpose had not been stretched, and when in the water were well freed of air bubbles by brushing. The results are certainly correct to the third decimal place.

The ratio of lateral lineal contraction to longitudinal dilatation was calculated from the formula $\sigma=\frac{e}{2r}-1$* on the assumption of the wires being isotropic. It seems evident, however, that the values of σ thus obtained cannot claim to be even approximately correct when the metal has been rendered very hard by the process of drawing, as was the case with copper (1), platinum-silver (1), brass (1), brass (2), and German-silver (2): here we meet with apparently impossible results.

The mean value of σ for the *different* substances† employed in the *annealed* condition $=\cdot 2515$, a number closely according with that assigned by Poisson as the value of σ for *each*.

The metals copper (1), copper (2), platinum, aluminium, silver, and platinum-silver were obtained from Messrs. Johnson, Matthey, and Co. as chemically pure, and the zinc, lead, and tin wires as being as pure as could be got by the ordinary process of distillation.

Elasticity of Volume.

If e denote the value of "Young's modulus," and σ the ratio of lateral contraction to longitudinal extension, it can easily be proved that the elasticity of volume $=\frac{1}{3}\cdot\frac{e}{1-2\sigma}$, and as e in Table II. is measured in grammes per square centimetre, it follows that the increase of volume per unit resulting from a longitudinal stress of 1 grm. per square centimetre $=\frac{1-2\sigma}{e}$.

In the following table are given the values of the volume elasticity, which will be denoted by v, and of the alteration of volume $\frac{1}{3v}$ produced by the above stress.

In the same table, in order to complete the information given in Table II., is recorded the section of each wire in square centimetres; the section of the hard-drawn metals in Table II. being approximately equal to those given here for the annealed wires.

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* Thomson and Tait's Nat. Phil, p. 521.

† Copper (2) is not included in this estimate, as I have reason to believe that it was imperfectly annealed.

TABLE III.

Name of metal	Elasticity of volume $= v = \frac{e}{3(1-2\sigma)}$.	Alteration of volume produced by longitudinal stress of 1 grm. per square centim $= \frac{1}{3v}$.	Section in square centims.
Iron	1508×10^6	$221{\cdot}1 \times 10^{-12}$	6550×10^{-6}
Platinum (1)	$585{\cdot}7 \times 10^6$	$569{\cdot}0 \times 10^{-12}$	5178×10^{-6}
German-silver (1). . . .	1524×10^6	$218{\cdot}7 \times 10^{-12}$	5731×10^{-6}
Copper (1)	1045×10^6	$319{\cdot}0 \times 10^{-12}$	7310×10^{-6}
Copper (3) . . .	$920{\cdot}3 \times 10^6$	$362{\cdot}3 \times 10^{-12}$	18330×10^{-6}
Platinum-silver (1) . . .	2190×10^6	$152{\cdot}2 \times 10^{-12}$	7681×10^{-6}
Zinc (1)*	$348{\cdot}3 \times 10^6$	$957{\cdot}0 \times 10^{-12}$	8144×10^{-6}
Silver (1)	$930{\cdot}3 \times 10^6$	$358{\cdot}3 \times 10^{-12}$	5464×10^{-6}
Aluminium (1)	$316{\cdot}0 \times 10^6$	$1055{\cdot}0 \times 10^{-12}$	8632×10^{-6}
Tin (1)	$130{\cdot}1 \times 10^6$	$2562{\cdot}3 \times 10^{-12}$	7758×10^{-6}
Lead (1).	$76{\cdot}5 \times 10^6$	$4360{\cdot}0 \times 10^{-12}$	7374×10^{-6}

Remarks on Table III.

There is little to be said with reference to this table except to call attention to the great alteration which takes place in the order of several of the metals, with reference to their elasticity of volume, and that occupied by them in the tables of "YOUNG'S modulus." We find, for instance, platinum, which in the latter table stands second on the list of annealed metals, here ranking as seventh, whilst the alloys, platinum-silver and German-silver, are both higher than iron, the former of the two alloys conspicuously so. It would seem, moreover, that either small reliance can be placed on the method of determining the ratio of lateral contraction to linear elongation from observations of the longitudinal elasticity and torsional rigidity, or else that the volume elasticity varies considerably with different specimens of the same metal; for instance, the mean value for the modulus of bulk elasticity in the case of the two specimens of annealed copper recorded in the last table is 982×10^6, whereas THOMSON† gives the corresponding value for copper as 1717×10^6.

PERMANENT ALTERATION OF DENSITY PRODUCED BY LONGITUDINAL TRACTION.‡

A few experiments were made with a view to determine the *permanent* alteration of density which can be produced by longitudinal traction. Two methods were adopted:

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* Zinc, tin, and lead are, though in the drawn condition, added to this list, as the process of drawing had not hardened them in any degree sufficient to make much difference in either the torsional rigidity or the modulus of elasticity.

† 'Brit. Encyc.,' Art.: "Elasticity," Table I.

‡ For observations on the alteration of density produced by torsion and hammering, see Part II.

in the one the wire was lengthened by successive loads put on for three minutes each and then removed; in the other the wire was stretched by the hand or by the aid of a lever by equal amounts each time until breaking ensued. The balance used in determining the density was an exceedingly good instrument made for me by OERTLING for the purpose of measuring the coefficients of thermal expansion of the metals by weighing them in water at different temperatures. It will suffice here to state that it was possible with this instrument to weigh an object in water to $\frac{1}{10}$th of a milligramme. The air bubbles clinging to the sides of the metals were carefully brushed off, as it was not possible to boil them off, for fear of partially annealing the wires, and the proper corrections were made for temperature and air displaced. The following experiments serve to illustrate the general nature of the results obtained by the two methods:—

Experiment XXI.

SILVER (1).

Load in kilogs used in producing extension.	Specific gravity water at 4° C. = 1.	Total percentage of increase of length $= dl$	Total percentage of decrease of specific gravity $= d\Delta$.	$\frac{d\Delta}{dl}$.
1	10·47691			
6	10·47561	1·15	·0124	·0108
7	10·47207	3·64	·0461	·0127
7·75	10·46754	7·28	·0892	·0123
8·25*	10·46153	9·38	·1465	·0156

Experiment XXII.

COPPER (1).

Load in measures of water, each measure = 2·5 kilogs.	Specific gravity water at 4° C. = 1.	Total percentage of increase of length $= dl$.	Total percentage of decrease of specific gravity $= d\Delta$.	$\frac{d\Delta}{dl}$.
5	8·8252			
9	8·8251	2·58	·00113	·00044
10	8·8247	4·21	·00566	·00134
11	8·8102	6·94	·1699	·02448
12	8·8076	10·74	·1993	·01856
13	8·7968	16·27	·3216	·01977

Wire broken.

Experiment XXIII.

IRON.

Specific gravity water at 4° C. = 1.	Total percentage of increase of length $= dl$	Total percentage of decrease of specific gravity $= d\Delta$.	$\frac{d\Delta}{dl}$.
7·7849			
7·7771	9·35	·1002	·0107
7·7747	12·65	·1311	·0104
7·7730	16·87	·1529	·0091
7·7684	20·73	·2121	·0091
7·7520	25·41*	·4229	·0167

It will be observed that in all cases the ratio of the decrease of specific gravity to the increase of length at first increases to a maximum, then decreases, and again increases largely when the breaking strain has been reached. In any case, however, the alteration of density which can be produced by longitudinal traction is small, and in my own experiments never reached ½ per cent., though several of the wires were strained to breaking.

WERTHEIM has also obtained similar results.†

RELATION BETWEEN MODULI OF ELASTICITY AND INTERMOLECULAR DISTANCE.

If we denote the specific gravity of a substance by Δ, and A represent the atomic weight, the intermolecular distance will be proportional to $\left(\frac{A}{\Delta}\right)^{\frac{1}{3}}=\alpha$.

It is natural to suppose that as α diminishes the elasticity will increase, and in fact WERTHEIM has shown‡ that is the case, and moreover that approximately "YOUNG'S modulus" varies inversely as α^7.

In the next table will be found the products of $e\times\alpha^7$ and $r\times\alpha^7$ for the annealed metals.

TABLE IV.

Metal.	Specific gravity $=\Delta$.	Atomic weight A.	Intermolecular distance $=\left(\frac{A}{\Delta}\right)^{\frac{1}{3}}=\alpha$.	$e\times\alpha^7$.	$r\times\alpha^7$.
Iron (1)	7·759	56·0	1·932	1994×10^8	778×10^8
Platinum (1) . .	21·300	197·4	2·100	2688×10^8	1250×10^8
Copper (1) . . .	8·913	63·5	1·924	1133×10^8	430×10^8
Zinc (1)	7·138	65·0	2·088	1328×10^8	586×10^8
Silver (1)	10·491	108·0	2·175	1712×10^8	627×10^8
Aluminium (1)	2·732	27·5	2·159	1473×10^8	580×10^8
Tin (1)	7·264	118·0	2·533	1852×10^8	812×10^8
Lead (1)	11·193	207·0	2·644	1510×10^8	669×10^8

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* Wire broke.

† Ann de Chimie, 1844, tom xii.

‡ Ibid.

The mean values of $e \times \alpha^7$ and $r \times \alpha^7$ are respectively 1711×10^8 and 717×10^8, and as in WERTHEIM'S results, platinum and copper differ in respect to these products from the mean values, more than the other metals. I have ascertained, however, that there is greater accordance between both $e \times \alpha^7$ and $r \times \alpha^7$, in the case of the different metals, than can be obtained by taking the products of e and r with any other power of α.

THE INFLUENCE OF AN ELECTRIC CURRENT AND OF MAGNETISM ON THE TORSIONAL RIGIDITY OF METALS.

WERTHEIM has shown* that the longitudinal elasticity of metals is temporarily diminished by the passage of an electric current, independently of the alteration which would result from the elevation of temperature produced by the current: he has also proved that *long-continued* magnetization diminishes both temporarily and permanently the elasticity of iron and steel, but that if the magnetization be continued for only a short time there is no sensible effect. As it seemed desirable to supplement these observations by others on the torsional rigidity of metals, a few experiments were made with this object.

Experiment XXIV.

A copper wire, 8 feet in length and ·095 centim. in diameter, was suspended for observations of the torsional rigidity in the manner previously described. To the centre of the flat bar which carried the cylinder was soldered a piece of platinum wire, about 3 inches in length and ·05 centim. in diameter; the other extremity of this latter wire, which hung vertically downwards, dipped into a mercury cup, so that by means of the cup and a silk-covered copper wire soldered to the upper bar, from which was suspended the wire under examination, connexion could be made with a battery of GROVE'S cells, in the circuit of which was placed a GAUGAIN-HELMHOLTZ'S tangent galvanometer; with this galvanometer the current of one DANIELL'S cell freshly charged with sulphuric acid diluted with seven parts of water and sulphate of copper solution, a deflection of 24·9° was produced through a resistance of ·71 ohm.

Observations.	Time of one vibration.	Current	Total percentage of diminution of rigidity caused by the current.	Percentage decrease of rigidity produced by heating effect of current.	Percentage decrease of rigidity caused by the current proper.
	Seconds.				
Platinum wire out of mercury cup	5 570	0			
Platinum in cup	5·568	0			
Wire shortened	5·355	0			
	5·395	73°·4	1·54	·63	·91
	5·361	0	1·26	·63	·63

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* Ann. de Chimie, 1844, tom. xii., p. 610.

Experiment XXV.

Iron wire about 8 feet in length and ·13 centim. in diameter.

Observations	Time of vibration.	Current.	Total percentage of diminution of rigidity caused by the current.	Percentage of diminution of rigidity produced by heating effect of current.	Percentage of diminution of rigidity produced by the current proper.
	Seconds.				
	4·153	0°			
	4·197	65° 8	2·10	1·30	·80
	4·155	0°	2·00	1·30	·70
	4·207	68°·2	2 48	1·60	·88
After three days' rest . . .	4·180	0°			
	4·177	28° 5	·14—	·05	·19—
	4·191	46°	52	·25	·27
After three days' further rest . .	4 161	0°			
	4·252	71°	4 34	2 05	2 19
After three days' further rest .	4 164	0°			
	4 164	16° 3	0		

Experiment XXVI.

A piece of the same kind of iron wire as that used in the last experiment, about 35 centims. in length, was placed in the axis of a magnetizing helix, 30 centims. in length; the helix consisted of 1,200 turns of copper wire $\frac{1}{20}$th of an inch in diameter, wound round a glass tube of 3 centims. inner diameter and 3 centims. thickness.

Time of vibration.	Temperature of wire.	Current	Total percentage of diminution of rigidity caused by the magnetizing current.	Percentage decrease due to heating effect of the current.	Percentage decrease of rigidity due simply to magnetization.
Seconds.	° C.	°			
1 739	10	0			
1 743	16	80	46	28	·18
1 722	14	0	2 42	·07	2 35

Several other experiments of the same kind as the last were made with different magnetizing helices and current strengths, and all seemed to show that slight temporary diminution of rigidity is produced by high magnetizing force; but that magnetizing forces small in comparison with that indicated in the last table, and yet sufficient to cause very sensible magnetization, produced no sensible effect. The vibrations were counted for at least half an hour in each determination, and an initial arc of 10° of torsion was employed in all cases. The percentage decrease of rigidity produced by the rise of temperature caused by the current was calculated from KOHLRAUSCH'S formulæ,* for iron

$$n=n_0(1-\cdot000447t-\cdot00000052t^2),$$

and for copper

$$n=n_0(1-\cdot00052t-\cdot00000028t^2);$$

where n_0 and n represents the rigidity at 0° C. and t° C. respectively. In Experiment XXVI. a delicate thermometer was placed near the wire, half-way down the helix, and the rise of temperature calculated from the mean of several readings taken from time to time during the passage of the current. In Experiments XXIV. and XXV. the rise of temperature was determined in the following manner:—A current of 70° was passed through the copper wire used in Experiment XXIV., and the soldered junction of a thermo-element, made of fine German-silver and iron wires, each about 2 feet in length, was kept in close contact with the copper wire by placing the latter between the two former; the other ends of the wires forming the thermo-element were connected by means of silk-covered copper wire with the terminals of a galvanometer, and after being wrapped in tissue paper and cotton-wool, were placed in a clip-stand, which was drawn on one side so as to slightly press the soldered junction against the suspended copper wire, a layer of tissue paper having been used to insulate the latter from the former. After a short time the deflection of the galvanometer became constant, and on immersing the junction of the thermo-element in water, it was found necessary to raise the temperature 8·5° C. in order to produce the same deflection as before; 8·5° C. was therefore assumed to be the rise of temperature which would be caused in the copper wire by the above-mentioned current, and the rise of temperature produced by the other currents was calculated from the assumption that the heat generated would be proportional to the square of the current strength. In the case of the iron wire it was assumed that the specific resistance of the iron would be six times that of the copper, and the rise of temperature was calculated accordingly.

An examination of these last tables shows apparently that the torsional rigidity of copper and iron is temporarily decreased by the passage of a powerful current, but is very little altered by currents of moderate intensity.†

Experiment XXIV. also shows that the dipping of the platinum wire into the mercury-cup did not appreciably affect the time of vibration.

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* 'Brit. Encyc.,' Art.: "Elasticity," § 79.

† I cannot place so much confidence as I could wish in the results of these particular experiments, as far as the decrease of rigidity by powerful currents is concerned, the method employed for estimating the elevation of temperature produced by the current being evidently only calculated to give a very rough approximation to the true values. I hope, however, to be in a position at some future time to attack the question in an entirely different way.

CRITICAL POINTS.

Several determinations of the *permanent* increase of length produced by loading were made, and led to the discovery that in all *well and carefully* annealed metals there are at least two points at which a sudden change takes place in the ratio of the load to the permanent extension produced thereby; these points have been called critical points, and it appears that changes more or less profound take place in most if not all of the physical properties of the substance when these points are attained.

As, however, the subject will be fully discussed in Part II., where it will be shown how these points can be indirectly determined more accurately than in the ordinary manner, it will suffice here to say that the existence of the first of these critical points seems to prove beyond a doubt that there is *a true limit of elasticity* for each substance, and that this is intimately connected with the value of "YOUNG's modulus."

It would also appear[*] that at these two critical points sudden changes take place in the density of the substance.

SUMMARY OF PART I.

1. The magnitude of the temporary elongation which can be produced by any load on a wire which has experienced permanent extension is reduced by simply allowing the wire to rest either loaded or unloaded for some time after the permanent extension has taken place.

2. The length of the period of rest necessary to produce the maximum of the effect mentioned in 1 varies considerably with the nature of the metal of which the wire is made; with some metals a few minutes suffice, whilst with others, such as iron or steel, many hours are required.

3. The effect of rest mentioned in 1 is greater in proportion for large loads than for small ones, and apparently vanishes in the case of such small temporary alterations of length as are produced by causing the wire to vibrate longitudinally.

4. The magnitude of the temporary elongation which can be produced by any load on a wire which has suffered recent permanent extension is also reduced by heavily loading and unloading the wire, the rate of reduction diminishing with each loading and unloading.

5. A departure, as far as temporary elongation is concerned, from "HOOKE's law," more or less decided, always ensues after recent permanent extension, even when the weights employed to produce the temporary elongation do not exceed one-tenth of the breaking-load of the wire.

6. This departure is diminished very noticeably in the case of iron, and to a greater or less extent with all metals by allowing the wire to rest for some time either loaded or unloaded; it is also diminished by repeated loading and unloading.

7. With aluminium and zinc, and probably with the more viscous metals tin and

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[*] See Experiments XXI.–XXIII. inclusive.

lead, both the maximum temporary increase of length which can be produced by any load, and the complete recovery after the removal of the load, are only attained after an interval of several hours, provided that the weights used for the load be not very small compared with the breaking-load.

8. There is a small but decided departure from "HOOKE's law," as far as temporary elongation is concerned, in all cases where the load employed to produce the elongation is of moderate amount; this is the case even when sufficient rest has been allowed to enable the wire to attain its maximum elasticity.

9. We can therefore only obtain by the method of longitudinal vibrations values for "YOUNG's modulus," which are strictly comparable with those got by the method of static extension, by experimenting when we use the latter method with very small loads, and with the wire under the same conditions of stress and strain as those occurring when the former method is adopted.

10. In the case of all metals, permanent extension, if not carried beyond a certain limit, causes, whether rest is or is not allowed after the permanent extension has taken place, a diminution in the value of "YOUNG's modulus," as determined by the method of static extension.

11. If the permanent extension be carried beyond the above-mentioned limit, further permanent increase of length increases the value of "YOUNG's modulus."

12. The limit of permanent extension mentioned in 10 varies considerably with different metals, and with the time which is allowed to elapse after the permanent extension has taken place.

13. In the cases of iron, heavy loading for some time so increases the value of "YOUNG's modulus," as determined by the method of static extension, that even when the extension would have caused, without such loading, diminution of the modulus, this diminution can be changed to an increase; with copper this is not the case.

14. With iron wire which has been heavily loaded for some time, the ratio of the temporary elongation to the load producing it becomes less as the load employed becomes greater, until a certain limit, depending upon the extent of the previous heavy loading, has been reached; whereas with most other metals, and with iron which has suffered permanent extension without allowing the load which has produced the extension to remain for some hours on the wire, the elongation increases at first in greater proportion than the load.

15. The effects on the longitudinal elasticity, and on the torsional rigidity of steel, of suddenly chilling the metal after it has been raised above a bright red heat, are similar to those produced by excessive permanent traction.

16. The loss of torsional rigidity, which is caused by twisting or stretching a wire beyond the limits of elasticity, is diminished by rest.

17. The influence of rest mentioned in 16 is greater in proportion for large arcs of vibration than for small ones, and is more noticeable with iron and steel than with most of the other metals.

18. The loss of torsional rigidity mentioned in 16 is more sensible proportionally with large arcs of vibration than with small ones.

19. Continual vibrating through large arcs has a similar effect on the torsional rigidity to that produced on "YOUNG's modulus" by heavy loading and unloading.

20. The density of a wire is very little altered by permanent extension, even if the latter be carried to the extent of breaking the wire.

21. The values of "YOUNG's modulus" obtained for the different metals are, roughly speaking, inversely proportional to the seventh powers of the mean distances between adjacent molecules in these metals.

22. The torsional rigidity of copper and iron wires seems to be temporarily decreased by the passage of a powerful electric current through the wires.

23. The torsional rigidity of iron wire seems to be temporarily diminished when the wire is subjected to a powerful longitudinal magnetizing force.

24. The effects mentioned in 22 and 23 are apparently independent of any change produced by the electric current or the magnetizing force of the temperature of the wires.

25. There are, in every well-annealed wire, two critical points at which sudden changes take place in the ratio of the permanent extension produced by longitudinal stress which is gradually increased in amount and the magnitude of the stress.

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PART II.—ELECTRICAL CONDUCTIVITY.

THE TEMPORARY ALTERATION OF ELECTRICAL CONDUCTIVITY PRODUCED BY LONGITUDINAL TRACTION.

W. THOMSON, in 1856,* investigated the effects of tension on electrical conductivity in copper and iron wires, and, moreover, stated that he "had very nearly established, for the case of iron at least, that the augmented resistance due to tension, either temporary or permanent, is a very little more than can be accounted for by the change of form."

In 1877† I determined in absolute units the amount of alteration produced by longitudinal traction in the resistance of steel, iron, and brass, and proved that the temporary alteration of resistance resulting from increase of length and diminution of section is with iron and steel about two-fifths and with brass four-fifths of the whole observed change.

In the above-mentioned experiments I experienced considerable difficulty in obtaining accurate results in consequence of the minuteness of the changes to be measured, and, therefore, set about devising some plan whereby the variations of resistance caused by slight changes of temperature, which had proved a source of great annoyance, might be eliminated. In this attempt I have been entirely successful, and with the arrangements described below have determined, with I believe considerable accuracy, the very small changes of resistance which can be temporarily produced by mechanical tension.

Description of apparatus.

Pieces, about 7½ feet in length, of the same wires as used in Part I. were suspended in pairs, as in fig. 6‡, in an air chamber 4 feet in height, 4 inches inner diameter and 6 inches outer diameter; the inner of the two concentric cylinders of which the air chamber was composed being surrounded by a layer of water 1 inch thick enclosed between the two cylinders. This vessel, which was made of tinned iron, rested on a table, provided with a suitable aperture, R, and was furnished with two thermometers, T T, passing through the outer cylinder and into the axis of the inner one. The ends of the two wires were clamped into three short and stout brass blocks, A, B, C, which rested upon a support of hard wood, D D, each block being separated from its neighbours by wooden partitions. The block B was double the length of A and C, and into this was clamped one end of each of the wires, the other ends being clamped into A and C. A caoutchouc-covered copper wire connected a binding-screw on A with one

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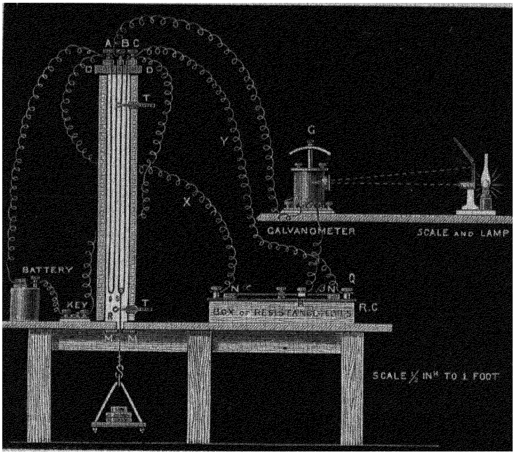
* Phil. Trans., Part IV., Feb. 28th, 1856, §§ 150–152.

† Proc. Roy. Soc., No. 183, 1877.

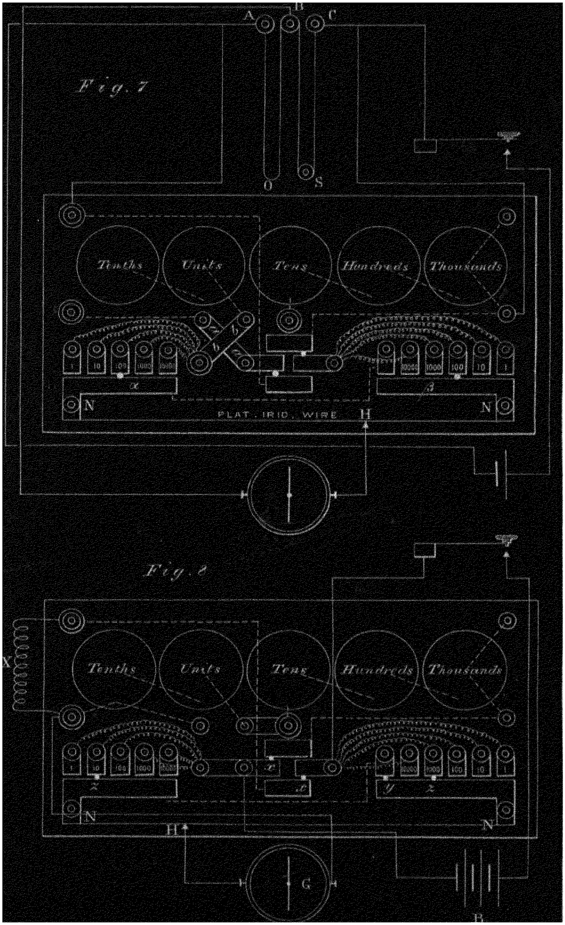
‡ In this figure the key employed to close the battery circuit is placed near the LECLANCHÉ cell and not in its actual position near the scale and lamp, in order to show the connexions more clearly.

Fig. 6.



pole of a large-sized Leclanché and a similar wire, X, connected the same binding-screw with a set of resistance-coils (α, fig. 7). In a similar manner C was connected with the other pole of the Leclanché, and through Y with another set of resistance-coils, β; whilst B, and the sliding-piece H, which traversed a wire, N N, made in the first instance of platinum-silver and afterwards of platino-iridium, uniting α and β, were joined to two terminals of a reflecting galvanometer, G. The wire to be strained was provided with a movable pulley, S, 2 inches in diameter, to which was attached by means of a stout wire the scale-pan used in Part I., or, as in some experiments, a large pail weighing 4 kilogs. and capable of containing about 60 kilogs. of water; both wires were, before suspension, surrounded with caoutchouc-tubing silk or other insulating material. The aperture, R, could be closed by two wooden shutters, M M, having small semicircular holes cut in the centre of the extremities adjacent to each other, so as to leave only just sufficient room for the stout wire to swing freely; moreover, still further to prevent any errors which might be caused by chance currents of air, the top of the air-chamber was well covered with baize after the wires had been adjusted, and the table was also surrounded on three sides with a like material.

Figs. 7 and 8.


A
B
C
Fig. 7
O
S
Tenths
Units
Tens
Hundreds
Thousands
1
10
100
1000
10000
b
b
α
β
N
PLAT. IRID. WIRE
H
N
Fig. 8
X
Tenths
Units
Tens
Hundreds
Thousands
x
x
z
y
z
N
H
N
G
B

It will be seen that the whole arrangement of wires, battery and galvanometer, forms, with the resistance coils α and β, a "WHEATSTONE'S balance," such that no part of the wires to be compared, namely, S and O, is out of the air-chamber. The galvanometer was provided with two sets of needles, which were astatic; the coil surrounding the upper set having a resistance of 7·487 ohms, and that surrounding the lower set a resistance of 4863 ohms, at the temperature 21° C. The wire of the former coil was divided into two equal parts, the ends of these being soldered to four terminal screws, so that, by connecting the two parts of the coil in "Multiple arc," the resistance could, if necessary, be diminished to 1·872 ohms, and the instrument rendered available for experiments requiring a galvanometer of low resistance. The upper coil only was used in this part of the enquiry, and yet the instrument proved so sensitive that, with the single cell employed for the current-motor, and with the adjusting magnet *assisting* the earth's directive force on the needles, a variation of conductivity of 1 part in 100,000* could be readily detected even in the most unfavourable case, which was that of a copper wire having a resistance of only ·0224 ohm; whilst in the majority of instances resistances were *measured within five parts in one million.*

The resistance coils were of platinum-silver, and the wire N N uniting α and β was, in the first few experiments, made of the same material, but was afterwards replaced by one of platinum-iridium; this latter wire was made expressly for me by Messrs. JOHNSON and MATTHEY, and seemed a marvel of accurate wire-drawing, as on testing it at various parts no variation of conductivity could be detected anywhere except at the two extreme ends, where it was firmly clamped into brass blocks; all the graduated portion of the wire, 470 millims. in length, seemed, within the limits of observation, to be perfectly uniform.

When required for measuring resistances in the ordinary way, the coils were arranged as in fig. 8; where X represents the wire under examination, G the galvanometer, B the battery, H the sliding-piece, and x, y, z conical plugs of brass with ebonite heads. Immediately to the right of X are a set of resistance-coils of the "dial pattern," ranging from $\frac{1}{10}$th of an ohm to 10,000 ohms; whilst below these are two sets of resistances, the arrangement and magnitude of which are sufficiently shown in the diagram; the plugs x, x serve to commute the position of X in the bridge, and the plug y to throw out, if necessary, the resistance N N, the dotted lines representing wires of extremely small resistance.

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* These beautiful instruments might, if necessary, be rendered even much more sensitive than they are by adopting a finer suspension. I have found that *a single fibre of unspun silk* is quite sufficient to sustain a weight equal to that on the suspension of many of the galvanometers which are now made, and anyone who will take the trouble to test practically, as I have done, the alteration which can be effected in the instrument by the *careful* selection of such a fibre will be surprised at the result. I feel little hesitation in asserting that my own galvanometer could be rendered *three* times as sensitive as it is now by such means; as it is, the silk suspension has been slowly untwisting for upwards of *seven years*, and still continues to do so. I have also found that keeping the galvanometer perfectly stationary, after it has once been placed in position, materially assists in preserving the magnetism of the needles.

For the purpose of this particular part of the enquiry, arrangements were made as in fig. 7, where it will be seen that the crossed pieces a a, b b, which are insulated from each other, serve to connect the tenths and units with the resistance-coils α.

Examination of the resistance-coils and of the platinum-iridium wire.

In order to test the uniformity of the platinum-iridium wire N N, two platinum-silver wires, of resistances 9 and 10 ohms respectively, were placed in the positions of O and S, fig. 7.

β was supplemented by a rheochord, and at both α and β the plugs were inserted in the 10 ohms resistance, whilst the rheochord was employed to bring H to the extreme right-hand end of the graduated portion of the wire. If, then, r denote the resistance of the ungraduated portion of N N on the right of H, together with the interposed portion of the rheochord and Y, whilst l denotes the resistance of the ungraduated part of N N on the left of H together with X, we have

$$\frac{10+l+(470-n_1)x}{10+r+n_1.x}=\frac{\text{resistance of O}}{\text{resistance of S}}=k \text{ say,}$$

or

$$10+l+(470-n_1)x=k(10+r+n_1.x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x is the average resistance of one division of N N, and n_1 is the number of divisions from the right-hand end of the graduated wire at which H is placed.

A resistance of $\frac{1}{10}$th ohm is now added to α from the tenths' dial-plate, and in consequence H has to be moved to n_2 divisions from the right-hand end in order to restore the balance;

$$\therefore\ 10{\cdot}1+l+(470-n_2)x=k(10+r+n_2.x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from (1) and (2) we obtain

$$x=\frac{{\cdot}1}{(k+1)(n_2-n_1)}.$$

Again the rheochord is adjusted so as to bring H about 10 millims. still further away from the right to, say, n_4 divisions, and on the removal of the $\frac{1}{10}$th ohm resistance H is brought back towards the right to, say, n_3 divisions; then as before

$$x=\frac{{\cdot}1}{(k+1)(n_4-n_3)}.$$

If the wire is uniform the value of x in the latter case should be equal to that of x in the former, and therefore

$$n_4-n_3=n_2-n_1$$

This was found to be so exactly the case that though H was provided with a vernier reading to $\frac{1}{10}$th of a millimetre, and the addition of the $\frac{1}{10}$th ohm required H to be shifted through about 300 millims., no difference whatever could be detected between $n_4 - n_3$ and $n_2 - n_1$, the galvanometer being sufficiently delicate to show an alteration of resistance equal to that of $\frac{1}{10}$th of a millimetre of N N. In the same manner each portion of the wire was tested, and it was concluded that no difference of resistance amounting to 1 in 3000 existed in any part of the graduated wire.

The uniformity of N N having been established, it was easy to compare the tenths with each other, and by determining the exact value of k, to find the values of their resistances in terms of the divisions of the platinum-iridium wire: this was accordingly done, and the values thus obtained were ascertained to accord fairly well with each other.

In order to secure still greater accuracy, the wire N N and the whole of the resistance-coils were afterwards tested by means of a second box of resistance-coils,* some eight or nine entire days having been spent in this work. These fresh trials confirmed the results obtained in the previous ones as regards the uniformity of N N and the *relative* values of the tenths; but the *absolute* values of the latter in terms of the divisions of N N were found to vary slightly on different days, as also did those of the rest of the resistance-coils. Thus the sum of the ten resistances in the units dial-plate were found on three separate days to be equal to that of 58,605, 58,739, and 58,437 divisions of N N respectively.†

From these and other experiments the average value of the resistances of each of the units at 17° C, at which temperature a unit agreed, according to KIESER, with 1 ohm, was equal to the resistance of 5859·4 millim. divisions of the platinum-iridium wire.

The units accorded very well with each other and with all the other resistances in the box except the tenths, nearly all the latter being slightly too low. The exact values, however, of each of the units and tenths were tabulated and used in calculating the results of the different experiments made in this and subsequent parts of the enquiry.

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* This box was kindly lent to me by Mr. KIESER, of ELLIOTT Bros., to whom I am also indebted for diagrams 7 and 8.

† These variations are certainly not due to errors of observation, as the results of trials made within one or two hours of each other agreed much more closely with each other, they may be attributed almost entirely to the prevalent plan of embedding resistance-coils in solid paraffin, whereby the temperature of the coils inside the box is frequently very different from that of the air outside: this plan is, I am convinced, a very bad one when great accuracy is required, as not only do the wires get heated to an extent which is very appreciable, even when only a single cell is employed, but a "PELTIER effect" is produced at the junctions of the coils and the brass-blocks which can never be properly got rid of; and thus the labour and care bestowed on these resistances are to a great extent lost. It would be better to fill the box with some such liquid as paraffin oil

Preliminary trials and final method of experimenting.

In the first attempts which were made the galvanometer was put into circuit by means of the usual contact-piece of the sliding-block H, immediately after closing the battery; but it was soon found that the very act of pressing down the contact-piece generated small thermo-electric currents* which, lasting some minutes, frustrated all attempts at making such accurate measurements as it was hoped would ultimately be obtained. H was therefore clamped by means of a suitable spring-and-catch, with which it was provided, to any desired part of the platinum-iridium wire, and a double key was employed by means of which first the battery and then immediately afterwards the galvanometer were put into the "bridge." Here again, however, exactly the same difficulty was encountered, and by no device of covering the hand and key with cloth or silk could this source of error be entirely avoided. The double key was therefore discarded, and the galvanometer being always kept in the "bridge," a single key was used for closing the battery circuit for the brief space of time necessary for observing whether this act caused any difference of potential at the terminals of the galvanometer.

Of course the zero-point of the galvanometer needles was continually being altered by the thermo-electric currents produced by the frequent shifting and clamping of H, but this circumstance did not affect the results, and though in the case of iron a slight trouble was experienced sometimes from the "kick" of the needles due to circular magnetization, this difficulty was after a few trials surmounted, and from this point the measurements proceeded very satisfactorily.

In most instances α and β were made of about 100 ohms' resistance, and though such large resistances were, it is true, out of proportion to those of the other branches of the bridge, yet, as has been already observed, the arrangement proved of amply sufficient delicacy, and moreover rendered it impossible that any slight variations of the resistances of X and Y, which were each ·042 ohm, and of the wire N N, arising from changes of temperature, should cause any appreciable error; indeed, one great advantage of this method is that the galvanometer and the resistance-coils may be a hundred yards or more from the rest of the "bridge" without any chance of fluctuations of temperature materially influencing the result, even when the most minute variations of electrical conductivity are to be measured.

The deflections of the galvanometer were read in the usual manner by means of the image of a fine wire fixed vertically across one end of a small blackened tube, into the other end of which was fitted a lens for focussing the image of the wire on to the mirror of the galvanometer; and the end of the tube at which the wire was situated was illuminated by a paraffin lamp, placed so that the edge of the flame was in front of the wire, the reflected circle of light with the fine dark line across the centre being very clearly defined on the scale, though the latter was at a distance of 6 feet from the galvanometer.*

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* This fact has, I find, been also noticed by GLAZEBROOK, Phil. Mag., April, 1881, No. 68.

The adjusting magnet was almost always used to *assist* the directive force of the earth's magnetism on the needles, as it was found that by so doing the shifting of the zero-point caused by the above-mentioned thermo-electric currents was considerably diminished, and at the same time the instrument was sufficiently sensitive. The wires S and O, which were made as nearly as possible of the same resistance, were, after being suspended, allowed to remain in the air-chamber for some time; β was then made 100 ohms and α adjusted, first by the resistances in the box as far as $\frac{1}{10}$th of an ohm, and finally by using the sliding-block H until no deflection of the image of the fine wire could be detected on closing the battery circuit.

As the needles of the galvanometer soon came to rest, and H could be very readily clamped and unclamped, it was possible to make the observations quickly; an interval of one minute was, however, generally allowed to elapse between two consecutive readings, as, though the battery-power was small and S and O of the same material and of the same section, yet in some cases the current evidently produced unequal heating effects in the two wires. Nor is this to be wondered at, as a difference of temperature of less than $\frac{1}{1000}$° C. would produce a sensible difference of resistance.

In ascertaining the temporary alteration of resistance caused by longitudinal traction, it was deemed advisable to adopt the same precautions as were used in determining the modulus of elasticity, as though the wires employed had been previously strained for the latter purpose and then allowed to rest for some weeks; they seemed, in some cases at least, to have become partially annealed, and it was found that the temporary alteration of electrical conductivity caused by loading was affected in precisely the same manner as the elasticity, by stress producing recent permanent extension.

Great care was taken in loading and unloading the wire S, and in the experiments which were made with the first two or three wires a large but light pail was used instead of the scale-pan. Into this vessel measured amounts of water were allowed to flow slowly through a piece of caoutchouc tubing, when it was necessary to apply stress, and the unloading was accomplished by suffering the same water to pass gently out through another piece of tubing connected with an orifice at the bottom of the pail. During the loading this latter tube was hitched up by the side of the vessel. It was found, however, that with practice quite as accurate results could be obtained by using a scale-pan in the ordinary manner, and as time was thereby saved this method was finally adopted.

Formulæ employed.

The temporary alteration of resistance which was in any case produced was so small that it could be measured by the wire joining α to β. If, therefore, A and B denote the number of millimetre divisions of this wire which have a resistance equal to the branches α and β respectively, including in these the connecting wires X and Y and the parts of the divided wire on either side of H, n be the number of divisions through which it is necessary to move H in order to restore the balance of resistance when disturbed by a load W, and S be the section of the wire, then, denoting by x the increase per unit of resistance which would be produced by a unit load acting on unit area, we have within a sufficiently close degree of approximation

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* The distance of the lamp and scale from the galvanometer is not drawn to the same scale in fig. 6 as for the other arrangements, for the purpose of avoiding the taking up of too much space.

$$x=\frac{(A+B)\times n\times s}{A\times B\times W}.$$

Again, if x be multiplied by e we obtain the alteration of resistance per unit which would result from doubling the length of the wire by the application of longitudinal traction; therefore denoting this latter value by y, we have

$$y=e\times x.$$

Part of y is due to mere increase of length and diminution of section; this part $=1+2\sigma$ very nearly. Thus the alteration of the specific resistance produced by the traction

$$=y-(1+2\sigma).$$

It will be seen that x and y are calculated on the assumptions that the change of resistance is directly proportional to the stress and also to the elongation; both these assumptions were found to be nearly correct, but neither are strictly so.

The following experiment out of many will serve to show (a) that the temporary alteration of resistance is nearly but not quite proportional to the load, and (b) that it is possible to measure with considerable accuracy minute changes of electrical conductivity even when the resistance of the wire used is small.

Experiment I.

An annealed copper wire ·154 centim. in diameter and having a resistance of only ·0224 ohm was loaded by pouring 12 measures of water, each having a weight of 5825 grms., into a pail attached to the pulley on the wire. This load, which was four-fifths of the "breaking-weight," was suffered to remain on the wire for some hours and was then removed. The following consecutive observations were begun next day and extended over three days, two trials being made on each.

The numbers in the column headed "Temporary alteration of resistance" are the divisions of the platinum-silver wire N N, through which it was necessary to shift H in order to balance the effect of the load.

Number of trial.	Number of measures employed for the load.	Temporary alteration of resistance.
I.	3	188·0
	6	383·0
	9	584·0
II.	3	189·0
	6	381·0
	9	581·0
III.	3	188·0
	6	384·5
	9	583·0
IV.	3	189·5
	6	384·0
	9	583·5
V.	3	188·0
	6	385·0
	9	583·0
VI	3	190·0
	6	384·0
	9	584·0

The mean values for three, six, and nine measures are respectively 188·8, 383·6, and 583·1, and none of the observations differ from these mean values by ·7 per cent.

Thus for the first three measures we obtain a mean alteration of 188·8, for the second 194·8, and for the third 199·5.

Experiment II.

The same wire as in the last experiment, after having been repeatedly loaded with 13 measures, was tested with the same weights as before with the following results:—

Load in measures.	Temporary alteration of resistance.	Average alteration per measure.	Difference between consecutive averages.
3	207·0	69·00	..
6	416·7	69·45	·45
9	629·4	69·93	·48

Experiment III.

A piece of platinum (1) annealed, which had been repeatedly loaded with 12 kilogs.

Number of kilogs. in load.	Temporary alteration of resistance	Average alteration per kilog.	Difference between consecutive averages.
4	108·5	27·13	..
6	163·5	27·25	·12
8	219·0	27·38	·13
10	277·0	27·70	·32
12	339·0	28·25	·55

Experiment IV.

The same wire as in the last experiment, which had been further loaded and unloaded with 15 kilogs. until the recovery had become perfect for this load.

Number of kilogs. in load.	Temporary alteration of resistance.	Average alteration per kilog.	Difference between consecutive averages.
1	27·5	27·50	
3	83·0	27·67	17
5	139·0	27 80	13
7	195 5	27 93	·13
9	252·5	28 06	·13
11	310 0	28 18	·12
13	368 0	28 31	13
15	430 0	28 67	·36

Experiment V.

A piece of silver (1) annealed, which had been previously loaded and unloaded with 8 kilogs. until the recovery had become perfect.

Number of kilogs. in load.	Temporary alteration of resistance	Average alteration per kilog.	Difference between consecutive averages.
2	94	47 00	
4	189	47 25	25
6	285	47·50	25
8	383	47·88	38

Experiment VI.

A piece of platinum-silver (1) rendered very hard by drawing, and which had been loaded and unloaded several times with 20 kilogs.

Number of kilogs. in load.	Temporary alteration of resistance.	Average alteration per kilog.	Difference between consecutive averages.
4	79 00	19·75	
8	161·80	20 23	48
12	246·25	20·52	·29
16	330 00	20·63	·11

Experiment VII.

An annealed iron wire which had been loaded for 24 hours with 30 kilogs.

Number of kilogs. in load.	Temporary alteration of resistance.	Average alteration per 2 kilogs.	Difference between consecutive averages.
2	9·0	9 00	
4	20 5	10·25	1·25
6	32·2	10·73	·48
8	44·0	11·00	·27
10	56·3	11·26	26
12	68·5	11·42	16
14	80·8	11 54	·12
16	92·9	11·61	07
18	105·1	11 68	07
20	117·4	11 74	06
22	129·8	11·80	06
24	142·3	11·858	058
26	155·0	11·92	062

Remarks on Experiments I.–VII. inclusive.

All these experiments prove that the ratio of the resistance-increase to the load is not quite constant, but that the former increases in a greater proportion than the latter. This want of proportionality is seen from Experiments I.–IV. inclusive to be materially diminished, though never entirely made to vanish, by repeated heavy loading and unloading, and it appears impossible to find any single formula which will express exactly the relation between the load and the alteration of resistance for all conditions of the wire; consequently the values of x given in Table I. are calculated from observations made with two weights, the larger never exceeding one-fourth of the "breaking-weight," and being twice the smaller; and the mean of the two numbers obtained by dividing the observed alteration by the load is taken to represent the change produced by unit load, this mean being in no case different from either of the numbers by more than 1 per cent.

The columns in which are recorded the differences between the consecutive average alteration per kilogramme show that when the load employed in determining the temporary alteration of resistance approaches closely to the highest load which has been used in the preliminary operations, a comparatively rapid increase takes place in the ratio of the temporary alteration of resistance to the load.

From Experiments VI. and VII. we learn that in the case of a metal which has been rendered very hard by the process of drawing, or in the case of iron* which has been subjected to a heavy load for some time, the average alteration of resistance per unit load increases less and less rapidly up to a certain degree of stress, and afterwards begins to increase more and more rapidly, whereas with the annealed metals the average alteration increases at first by almost equal amounts.

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* Probably also in the case of any metal possessing a coercive force comparable with that of iron.

Experiment VIII.

A piece of platinum (1) annealed was loaded and unloaded several times with 12 kilogs., and was immediately afterwards tested, with the following results:—

No. of kilogs. in load.	Temporary alteration of resistance.	Average alteration per kilog.	Difference between consecutive averages.
2	66·5	66·5	
4	134·0	67·0	·5
8	276·0	69·0	2·0
After a rest of 20 hours with all stress removed except that of scale-pan.			
2	67·0	67·0	
4	136·0	68·0	1·0
8	281·0	70·25	2·25

From the last experiment it is evident that part of the increase of elasticity which is gained by repeated heavy loading and unloading is lost by allowing the wire to rest, and, moreover, that the departure from "HOOKE's law" which we have seen to be appreciably lessened under the influence of the former cause is increased again by the latter.

In Table I. will be found the values of the specific resistance of the different metals, the increase of resistance per unit which is temporarily produced by a stress of 1 grm. per square centimetre, the increase of resistance per unit which would be caused by stress sufficing to double the length of the wire, and the increase of specific resistance per unit which would be caused by stress sufficing to double the length of the wire; the specific gravity and section of the wire are approximately the same as those recorded in Part I.

TABLE I.

Name of metal.	Condition.	Specific resistance at 12° C., *i.e.*, resistance in ohms of 1 cubic centim., between opposing faces = R.	Increase of resistance per unit produced by stress of 1 grm per square centim. $= x$.	Increase of resistance per unit which would be caused by stress sufficing to double the length of the wire $= y$.	Increase per unit of specific resistance which would be caused by stress sufficing to double the length of the wire $= z$.
Iron (1)	Annealed	1074×10^{-8}	2111×10^{-12}	4·180	2·618
Iron (2)*	Hard drawn	1217×10^{-8}	2100×10^{-12}	4·289	2·639
Iron (3)	Hard drawn	1201×10^{-8}	2197×10^{-12}	3·699	2·057
Piano steel (1)	Hard drawn	1653×10^{-8}	1910×10^{-12}	3·619	2·081
Piano steel (2)	Hard drawn	1882×10^{-8}	1831×10^{-12}	3·602	2·084
Platinum (1)	Hard drawn		2233×10^{-12}	3·341	2·239
Platinum (1)	Annealed	1434×10^{-8}	2285×10^{-12}	3·404	2·252
German-silver (1)	Annealed	2830×10^{-8}	1501×10^{-12}	1·937	0·107
German-silver (2)	Annealed		1545×10^{-12}	2·099	0·345
German-silver (1)	Hard drawn	2713×10^{-8}	1892×10^{-12}	2·138	0·138
Copper (1)	Annealed	$240·8 \times 10^{-8}$	2210×10^{-12}	2·564	0·934
Copper (1)	Hard drawn	$244·0 \times 10^{-8}$	1988×10^{-12}	2·880	
Copper (3)	Annealed	$201·4 \times 10^{-8}$	2324×10^{-12}	2·656	1·070
Copper (2)	Annealed	$187·3 \times 10^{-8}$	2396×10^{-12}	2·918	1·012
Platinum-silver (1)	Annealed	3236×10^{-8}	2346×10^{-12}	2·464	0·624
Platinum-silver (1)	Hard drawn	3127×10^{-8}	2437×10^{-12}	2·530	
Brass (1)	Hard drawn	834×10^{-8}	2302×10^{-12}	2·275	0·101
Brass (2)	Hard drawn	$656·7 \times 10^{-8}$	2229×10^{-12}	2·203	0·231
Zinc (1)	Hard drawn		4406×10^{-12}	3·379	2·113
Silver (1)	Annealed	$161·9 \times 10^{-8}$	4272×10^{-12}	3·851	1·531
Silver (1)	Hard drawn	$173·7 \times 10^{-8}$	4561×10^{-12}	3·487	1·703
Aluminium (1)	Hard drawn	$316·7 \times 10^{-8}$	1883×10^{-12}	1·260	−0·420
Aluminium (1)	Annealed	$311·3 \times 10^{-8}$	1896×10^{-12}	1·276	−0·262
Tin (1)	Drawn	1166×10^{-8}	10546×10^{-12}	2·920	1·630
Lead (1)	Drawn	2142×10^{-8}	17310×10^{-12}	2·885	1·613

Remarks on Table I.

It will be seen from the above table that the specific resistance of iron is more increased by a given amount of elongation than that of any of the other metals, and that the specific resistance of aluminium is actually *decreased* by stress in the line of flow of the current; this latter fact being signified by a minus sign placed opposite aluminium in the sixth column.

It is also remarkable that the value of z for the alloys, platinum-silver, German-silver and brass should be considerably less than that of their components; and this circumstance, taken in connexion with the comparatively large increase of resistance of iron, would suggest that there is some relation between increase of electrical resistance caused by rise of temperature and that due to mechanical stress.

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* Iron (2), iron (3), steel (1), steel (2), and brass (2) were tested some years ago by a method similar to the one here described.

But the change of resistance attending a given amount of expansion caused by rise of temperature is, in the case of iron, more than a hundred times that resulting from the same amount of expansion produced by mechanical stress; and it is evident that with all metals the alteration of electrical conductivity following on alteration of temperature is due for the most part to other causes than mere contraction and expansion.

Carbon.

As it seemed desirable to extend these researches to as many substances as possible, some experiments were made with carbon rods such as are used for electric lighting purposes. These rods were between 40 and 50 centims. in length, and of different diameters, and their moduli of longitudinal elasticity could be readily determined by holding them in the centre and rubbing them along their length with a resined glove. The note obtained by the longitudinal vibrations, though, of course, very high in pitch, was quite clear and distinct, and very concordant results were obtained when the same pieces were tried at different times. The pitch of the note was determined by the syren, and the following experiment, taken at random out of my note book, will suffice to show what accuracy can be attained with this instrument* even with notes of very high pitch.

Experiment IX.

A carbon rod of length ·496 metre, rubbed longitudinally by means of a resined glove, gave a note, the lower double octave of which was taken on a monochord; the syren was then raised to the pitch of the monochord, and the number of vibrations counted for two minutes at a time.

Number of trial	Number of vibrations recorded by the syren in two minutes.
1	6860×20
2	6822×20
3	6835×20
Mean . .	$= 6839 \times 20$

In this experiment the monochord was re-tuned at each trial, and it will be observed that the mean value does not differ from any of those forming it by so much as $\frac{3}{10}$ per cent. Of course, by extending the time of each trial and the number of observations, still greater accuracy could have been obtained.

The formula employed for calculating the elasticity is $E = \frac{n^2 \times 4\, l^2 \times \Delta}{9810}$, in which l is

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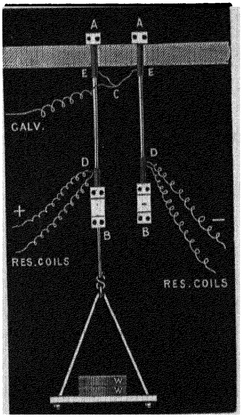
[*] I have to thank Mr. FURSE, the Curator of the Physical Museum at King's College, for his able assistance in this part of the work.

the length of the rod in metres, n the number of vibrations, Δ the density, and E the elasticity in kilogrammes per square millimetre.

Fig. 9.



In order to determine the alteration of resistance which could be produced by longitudinal traction, the rods were well coated for about 2 inches of their lengths with copper deposited by electrolysis, and were then arranged as shown in fig. 9. The rods passing through two holes in a table were secured at their upper ends by two clamps, A A, and at their lower extremities were fastened two other clamps, B B, either of which would also serve to carry a scale-pan suspended by an iron wire attached to its lower extremity. Two pieces of silk-covered copper wire, about 6 inches in length and $\frac{1}{30}$th of an inch in diameter, were soldered on to the deposited copper at E, and these being also joined together were connected with the galvanometer. Two other pairs of similar wire were soldered at D, and one of each pair was connected, as usual, with the resistance coils, and the other with one pole of the battery. The resistances of the silk-covered copper wires at E, which were small compared with those of the rods themselves, were calculated by determining the resistance of some 6 feet of the wire, and assuming that the resistances of the short pieces at E and the actually determined resistance of the longer piece were proportional to their lengths. The whole table was surrounded on all sides except one with baize, and the mode of proceeding the same as usual. Experiment shows that with the loads employed the alteration of resistance is nearly proportional to the load.

Experiment X.

Carbon rod, length between E and D 45 centims., diameter ·434 centim.

Number of kilogs in the load.	Number of divisions of the platino-iridium wire through which it was necessary to move the sliding piece in order to restore the balance.	Average alteration of resistance per kilog. in terms of the division of the platino-iridium wire.
2	37·00	18 50
4	74·75	18 69
6	111·80	18·63
		Mean 18 61

In Table II. will be found some data respecting the alteration of resistance produced by loading, the values of "YOUNG's modulus," specific resistance, specific gravity, and section of each rod.

TABLE II.

Number of carbon.	Section in square centims.	Specific gravity water at 4° C=1 $=\Delta$.	Specific resistance at 15° C.	"YOUNG's modulus" in grammes per square centim. $=e$.	Increase of resistance due to a load of 1 grm. per square centim. $=x$.	Increase of resistance which would be caused by a load sufficient to double the length $=e\times x$ $=y$.	Increase of specific resistance which would be caused by a load sufficient to double the length $=z$.
1	1411	1 518	4591×10^{-6}	$233\ 8\times 10^{6}$	11420×10^{-12}	2 470	0·970
2	·1307	1 550	4182×10^{-6}	$264{\cdot}3\times 10^{6}$	8991×10^{-12}	2·377	0·877
3	·1480	1·552	4483×10^{-6}	$270{\cdot}0\times 10^{6}$	9535×10^{-12}	2 575	1 075
4	·1351	1·579	3937×10^{-6}	$329\ 0\times 10^{6}$	8610×10^{-12}	2·835	1·335
5	·1216	1·581	3774×10^{-6}	$279{\cdot}0\times 10^{6}$	7684×10^{-12}	2·144	0·644
Means		1·556	4193×10^{-6}	$275{\cdot}2\times 10^{6}$	9248×10^{-12}	2 480	·980

Remarks on Table II.

The sections of the rods were much more uniform than would perhaps have been expected; for example, in the case of No. 3, as measured by a gauge at five points equally apart, the diameter was found to be ·434, ·434, ·434, ·434, and ·433 centim., and the other rods were found to be nearly of the same uniformity as this one.

The specific gravities were determined by breaking the rods into pieces about 5 inches in length and binding a thin copper wire round them, which latter was fastened to a very fine platinum wire, so that the pieces could be weighed in air and in water.

It should be stated that carbon in this form being porous, the specific gravity as usually reckoned will be found to increase if the substance be allowed to remain in water for any time, and especially so if the carbon be boiled in water. The following experiment will show the extent of alteration of the apparent specific gravity :—

Experiment XI.

Several pieces of carbon tied together with fine copper wire, and boiled for five minutes; the pieces then taken out of the hot water, kept for five minutes in cold water, and afterwards suspended by a fine platinum wire in a large vessel filled with water at 15° C.

Weight of carbon in water.	Time after immersion in the large vessel of water, at 15° C.
4 800	5 minutes
5·020	15 „
5·140	35 „
5·255	75 „
5 291	140 „

The pieces then boiled a second time for 40 minutes, and after having been kept in cold water for 30 minutes, again tested for loss of weight.

Weight of carbon in water	Time after immersion in the large vessel of water, at 15° C.
5 314	5 minutes
5 340	45 „
5 393	13 hours.
5 432	5½ days.
5 452	12½ „

As the temperature of the water varied very little during the periods of observation, we see that the apparent specific quantity went on increasing for more than 12 days, and that the total percentage of increase amounted to nearly 12. The specific gravities, however, recorded in the table are calculated from the loss of weight observed shortly after immersion in water, the pieces of carbon not having been boiled, and the section as determined by dividing the loss of weight by the length agreed very well with the section measured by the wire-gauge.

With the exception of number 5, the value of "YOUNG's modulus" increased very largely as the specific gravity increased; in fact, the value of e is roughly proportional to Δ^8.

Beetz[*] has also determined the values of e for several carbon rods, used for electric lighting, by longitudinal vibrations, and it would appear from the results obtained by him that $e \div \Delta^8$ is also fairly constant. The mean values of **Beetz**' results and my own are given in the next table.

TABLE III.

Specific gravity = Δ.	Value of e in grammes per square centim.	$e \div \Delta^8$.
	BELTZ	
1·532	$152·0 \times 10^6$	501×10^4
1·547	154·7 „	466 „
1·564	174·7 „	488 „
1·580	193·5 „	498 „
1·593	205·4 „	495 „
1·631	254·8 „	509 „
	Mean value of $e \div \Delta^8 = 493 \times 10^4$	
	TOMLINSON.	
1·518	$233·8 \times 10^6$	829×10^4
1·550	264·3 „	793 „
1·580	304·2 „	783 „
	Mean value of $e \div \Delta^8 = 802 \times 10^4$	

Though, however, $e \div \Delta^8$ is roughly a constant for different specimens of carbon by the same maker, the value of this constant may be evidently very different for samples from different makers, the mean value of **Beetz**' samples for $e \div \Delta^8$ being 493×10^4 and that of my own specimens 802×10^4.

The value of e given in Table II. for number 3 is calculated from the formula $e \div \Delta^8 = 802 \times 10^4$.

It will be noticed that the product $e \times x$, *i.e.*, the increase of resistance which would be caused by a load sufficient to double the length of the rod, is nearly a constant for the different specimens, and on referring to Table I. it will be observed that this product is much less than that for several of the metals, though the increase of resistance caused by a stress of 1 grm. per square centimetre is of course, in consequence of the small elasticity of the carbon, greater than any of the metals examined except tin and lead, whose elasticities approach much more nearly to that of this kind of carbon than any of the other substances.[†]

In calculating the alteration of specific resistance it was assumed that $\sigma = ·250$, but though the values given in the last column of Table II. are in consequence of the want of knowledge of σ approximate to the true ones, they must be sufficiently so to prove that not only is the total resistance of the carbon increased by loading but so also is the *specific resistance.*

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[*] Ann. der Phys. und Chem., 1881, No. 1, p. 67.

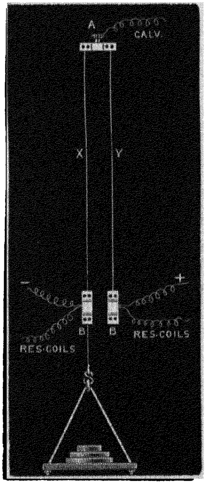
[†] The elasticity of the rods here examined is very nearly equal to that of tin.

The mean value of the specific resistance of the different specimens is 4193×10^{-6}, and is therefore more than 100 times greater that the corresponding number for platinum-silver, which latter metal ranks highest in the list of metals in Table I.

NICKEL.

Through the kindness of Messrs. JOHNSON and MATTHEY, who after some difficulty succeeded in drawing for me two pieces of almost pure nickel wire 8 feet in length, I was able to make some experiments on this metal, in which I obtained results so completely differing from those observed in the case of the other substances, that I may perhaps be excused for treating them in some detail.

Fig. 10.



The metal when first received was in the hard drawn condition, and in this state was tested for torsional rigidity and longitudinal elasticity, for the former in the usual manner and for the latter by longitudinal vibrations. The pieces were then suspended in the air chamber, and since it was not thought desirable to bend the wire, the following arrangements were made, which are sufficiently shown in fig. 10.

X and Y are the two wires to be compared, the clamp A rests as before on the top of the air chamber, and of the two clamps B B fastened to the lower ends of X and Y one has a stout iron wire attached to it for the purpose of a suspension for the scale-pan. Also to each of the clamps B are soldered two silk-covered copper wires $\frac{1}{40}$th of an inch in diameter and 3 feet in length, one of these serving to connect the clamp with one of the battery wires and the other with the caoutchouc-covered wire leading to the resistance coils. The wires X and Y were each 3½ feet in length, but were not cut off from the 8 feet pieces, the remainder of the wire* being allowed to hang down on the outside of the air chamber, whilst the silk-covered copper wires fastened to the clamps B B were brought down through the hole in the table at the bottom of the air-chamber. When the experiments on the hard-drawn wire had been finished, the metal was well annealed and similar ones made on it in this new state, with this difference, however—that now the wire X was tested with larger and larger loads until it finally broke. The results of the experiments made on the wire in the hard drawn and annealed conditions will be found in Table IV.

The curves in Table V., showing the temporary alteration of resistance produced in the annealed wire by different loads after the wire had been previously stretched for some time with weights of 16, 18, 20, 22, and 24 kilogs., are drawn with their abscissæ to represent the percentage alteration of resistance on a scale of ·0025 to the millimetre, and with their ordinates the number of kilogrammes per square centimetre in the load on a scale of 25 kilogs. to the millimetre.

TABLE IV.

Condition.	Section in square centims.	Specific gravity at 20° C., water at 4° C. = 1.	Specific resistance at 20° C.	Torsional rigidity in grms. per square centim. $= r$	"YOUNG'S modulus" in grms. per square centim. $= e$	Ratio of lateral contraction to elongation.
Hard drawn	005832	8·7066	1757×10^{-8}	$723·5 \times 10^{6}$	$2270·6 \times 10^{6}$	570
Annealed .	·005832	8·7386	1736×10^{-8}	$758·5 \times 10^{6}$	$2174·6 \times 10^{6}$	433

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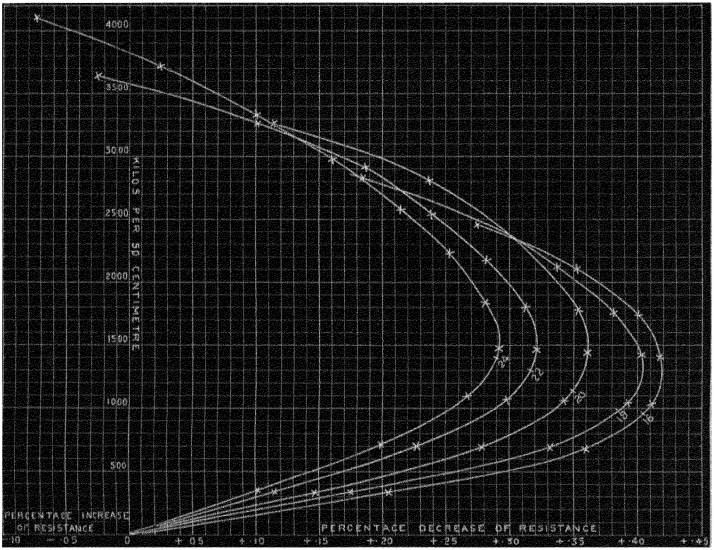
* Not shown in the figure.

TABLE V.—Curves showing the temporary alteration of electrical conductivity of nickel wire produced by longitudinal traction at the temperature 22° C.



Remarks on Tables IV. and V.

One of the first points to notice in Table IV. is the large value of "YOUNG's modulus," which for both the hard drawn and the annealed conditions of the nickel exceeds the highest values obtained for pianoforte steel wire.

Again we have seen in Part I. that the value of "YOUNG's modulus" increases as the mean distance between the molecules diminishes, and the ratio of the specific gravity to the atomic weight of nickel, which is inversely as the cube root of the mean distance between one molecule and another, is ·1493, whilst that of iron, which with copper has the ratio higher than is the case with any of the other metals mentioned in this paper, is ·1386. We have also seen that $e \times \alpha^7$ is roughly a constant for the different metals, where α is the mean distance between one molecule and another, and e is the value of "YOUNG's modulus;" in the cases of nickel and iron this product is $\cdot 1838 \times 10^{12}$ and $\cdot 1993 \times 10^{12}$ respectively.

The ratio of the lateral contraction to the linear elongation obtained from the formula $\sigma = \frac{e}{2r} - 1$ of the hard drawn nickel is of course impossible, as was the case, it will be remembered, with the value of σ with hard-drawn copper, brass, German-silver, and platinum-silver, and the wire therefore furnishes another proof that the formula cannot be applied to hard drawn metal.

The most remarkable feature, however, presented by nickel is that shown in Table V., where we learn that the resistance is absolutely *decreased* up to a certain extent of loading and then begins to increase. We see, moreover, that the *maximum decrease* becomes less and less as the wire receives more and more permanent extension, and that the point of loading where this maximum occurs gradually rises with the amount of permanent extension.* As might be supposed, therefore, the decreases of resistance obtained with two loads of 3 and 6 kilogs. were with the hard drawn metal much less than with the annealed one. Again, if we take the average decrease of resistance produced by a load of 1 grm. per square centimetre between the points of zero load and the load producing maximum decrease, we find it in the case of the outer curve to be 3216×10^{-12}, and the product of this number by $e = 6{\cdot}994$, whilst the decrease of *specific* resistance attending unit increase of length, or the number corresponding to z in Table I., would amount to 8·860. All these numbers, especially the last, are very considerably greater than the corresponding *increases* of resistance obtained with any of the other metals. We thus observe that whether we regard the peculiarity of loading up to a certain point producing decrease of resistance, and after this point increase, or the comparatively enormous temporary variations of resistance produced by loading, nickel stands by itself, and the idea at once suggested itself that this abnormal behaviour of the metal might be due to the influence of circular magnetization caused by the current employed in balancing the wire and the comparison-wire. Accordingly two experiments were made of the following nature:—First, as the alteration of resistance might be only apparent and due to the fact that VILLARI's shock-currents were not the same in the stretched and unstretched wires,† both the galvanometer circuit and the battery circuit were kept closed, and the position of the light on the scale noted with different stretching weights: the readings taken in this way gave alterations of resistance which were exactly the same as those obtained by the usual method. Secondly, as the alterations might be really those of resistance, but due to the fact that circular magnetization, might‡ cause an alteration of resistance in both iron and nickel, and unequally in the stretched and unstretched wires, the resistances in the bridge were so adjusted that currents of one-half and one-fourth respectively of the current which had previously passed through the wire in determining the resistance might now do so; but again the alterations of resistance were found to be the same as before.* Now if E denote the electromotive force of the LECLANCHÉ, x and y the resistances of the wire and the comparison-wire, α and β the resistances in the box, and B the resistance of the battery and connexions, the current in x will be proportional to $\frac{E}{B(1+\frac{x+y}{\alpha+\beta})+x+y}$, or since x and y were only ·44 ohm each, whilst α and β were 100 ohms each, the current in x will be nearly proportional to $\frac{E}{B+x+y}$.

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* The numbers 16, 18, 20, 22, 24, on the curves represent the load which had previously been on the wire before testing for the temporary effect of loading.

† I did not think this likely, as I had not been able to detect anything of the kind when iron wire was used.

‡ According to AUERBACH, but not according to experiments tried by myself.

E was nearly 1·5 volts, and $B+x+y$ in the case of the weakest current was made 10 ohms; therefore this current would in absolute measure of C.G.S. units be ·015. Unless, therefore, the maximum difference of alteration of resistance caused by circular magnetization in the stretched and unstretched wires had been reached by a still smaller current than this, we cannot regard the curious behaviour of nickel in respect to the effect of stress on its electrical conductivity as due to circular magnetization.

Again, another idea suggested itself, namely, that the wires being suspended vertically might cause the resistance of the stretched and unstretched wires to be altered unequally by the earth's vertical magnetic force, but this latter, it will be seen, is of too small intensity to have any effect at all comparable with that observed in the stretching; and, moreover, experiments subsequently made with the wire at right angles to the magnetic meridian, showed plainly that this was not the case. We must therefore regard the abnormal effect produced on the electrical resistance of nickel as not due to the earth's magnetic force.†

We have seen also that, in the case of other metals which have suffered permanent extension, rest increases the elasticity and *diminishes* the temporary *increase* of resistance which any load is capable of causing, and that with iron this effect is very apparent. It was therefore an interesting point to determine whether the *decrease* of resistance which moderate loading produces in nickel would be increased by rest. The next experiment shows that this is so.

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* Similar experiments with iron had also previously shown that with this metal there is no appreciable difference in the alteration of resistance produced by stretching when different current strengths are employed in the bridge.

† After these experiments I re-tried iron with a view to ascertain whether *very small loads* might not produce decrease of resistance, but found that the smallest load that caused any effect whatever produced as before increase of resistance.

Experiment XII.

An annealed nickel wire loaded and unloaded several times on three different occasions, with weights of 18, 20, and 22 kilogs. respectively, and then tested with equal or lesser loads.

Load in kilogs. put on and off several times $= \alpha$.	Load used in testing after $\alpha = \beta$.	Temporary alteration of resistance produced by β. − Decrease of resistance. + Increase of resistance.	Time after using α at which β was applied.
18	First 8	−89	½ hour
	Next 10		
	First 8	−95	1 "
	Next 10	+60	
	First 8	−99	24 hours
	Next 10	+57	
20	First 8	−79	½ hour
	Next 12	+71	
	First 8	−82·5	7 hours.
	Next 12	+67·5	
22	First 8	−72·0	½ hour.
	Next 12	+64·5	
	First 8	−73·5	12 hours.
	Next 12	+62·5	

We see here that rest produces a very appreciable effect, increasing the negative alteration of resistance and diminishing the positive alteration; but we notice also that as the loads α become larger and larger the influence of rest becomes less and less marked; and, lastly, that as α increases to nearly the breaking load of the wire the effects both of the moderate load to produce decrease and of the excessive load to produce increase become more and more equal, and if we turn to Table V. we see plainly that the points in the curves showing the position of maximum decrease of resistance, which at first become wider and wider apart, at length begin to close up.

All this is intelligible if we bear in mind what has been proved in Part I., namely, that the elasticity of all metals is diminished by permanent extension carried to a certain point, and beyond this point increased, and provided that we assume that *temporary** elongation without regard to the stress producing it causes increase of resistance. The question then naturally arises, do stress and the consequent temporary strain produce on the electrical conductivity of substances opposite effects?

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* Permanent elongation we shall find produces in most metals first increase of specific resistance and then decrease, but in iron and nickel first decrease and then increase.

INFLUENCE OF PERMANENT EXTENSION ON THE TEMPORARY ALTERATION OF RESISTANCE CAUSED BY LONGITUDINAL STRESS.

It has been asserted in Part I. that the elasticity of a wire is diminished by permanent extension not exceeding a certain limit. In order to furnish still further evidence in favour of this assertion some experiments were made on the influence of permanent extension on the temporary alteration of resistance produced by longitudinal stress.

Experiment XIII.

A piece of copper (3) was subjected to loads which were gradually increased, and after the removal of each load the temporary alteration of resistance was determined from the mean values got by testing with from one to five measures of water.

Number of measures used for the permanent extension.	Number proportionate to alteration of resistance of wire.	Difference between consecutive alterations of resistances.	Number proportionate to temporary alteration of resistance caused by stress of 1 grm. per square centim.	Difference between consecutive alterations of resistance.
5	860	. .	644	.
6	862	2	648	4
7	865	3	652	4
8	878	13	658	6
9	912	34	681	23
10	952	40	692	11
11	985	33	693	1

Experiment XIV.

A similar piece of wire was loaded with from 12 to 14 measures of water, and after the removal of each load tested in a similar manner to the above.

Number of measures used for the permanent extension.	Number proportionate to alteration of resistance of wire *	Difference between consecutive alterations of resistances.	Number proportionate to temporary alteration of resistance caused by stress of 1 grm per square centim.	Difference between consecutive alterations of resistance.
12	1288	.	421	
13	1418	130	416	−5
14	1574	156	409	−7

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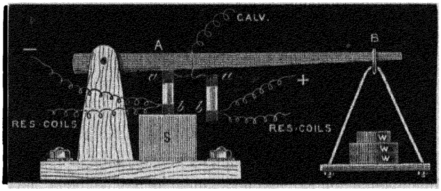
* Neither these numbers nor those in column 4 are on the same scale as the corresponding numbers in the last experiment.

The last two experiments show that with copper the temporary alterations of resistance caused by longitudinal stress increase up to a certain limit of permanent extension and then begin to decrease. We also see that the greatest changes in the values of the temporary alterations of resistance take place at those places where there are the greatest permanent changes of resistance.

THE EFFECT OF COMPRESSION ON ELECTRICAL CONDUCTIVITY.

It was anticipated that compression would in most cases produce decrease of electrical resistance, and as carbon seemed a suitable substance to experiment on, the following arrangement was made with the view of ascertaining whether compression would produce on the resistance of carbon the opposite effect to that caused by longitudinal extension :—S in fig. 11 is a block of stone on which rests upright the piece of carbon which it is desired to compress ; the other piece of carbon to serve as the comparison-piece is supported near the first by a clip-stand (not shown in the figure). The two pieces of carbon were each about 4 inches in length, and were well coated with copper deposited by electrolysis for a distance of about three-quarters of an inch at the ends. To the upper ends *a, a* were soldered two silk-covered copper wires as in the experiments on extension, and the junction of these was connected with the galvanometer, whilst to the copper deposited on the lower extremities *b, b* were soldered wires to serve for connecting with the battery and resistance coils. The compression was produced by putting weights into a scale-pan attached to the end B of a lever of hard wood, which in the position of the carbon rod produced on the latter a pressure five times that of the weights in the pan.

Fig. 11.



Experiment XV.

A piece of carbon rod, 4 inches in length and ·1386 square centim. in section, was compressed by putting a weight of 2 kilogs. on to the pan at the end of the lever.

Number of trial.	Alteration of resistance as represented by the number of divisions of the platino-iridium wire through which it was necessary to move the sliding-piece in order to balance the effect of putting on or taking off 2 kilogs.
1	118 0
2	118 5
3	110 0
4	115 5
Mean	115·4

From the results of Experiment XV. it was calculated that a compression of 1 grm. per square centimetre would cause a *decrease* of resistance of 6398×10^{-12} per unit, and as the longitudinal elasticity was $267{\cdot}2 \times 10^{6}$, the alteration of resistance attending an amount of compression which would suffice to halve the length of the rod would be 1·710, and of this alteration ·210 would be in the specific resistance of the rod. The specific resistance of the specimen at 15° C. was 4214×10^{-6}.

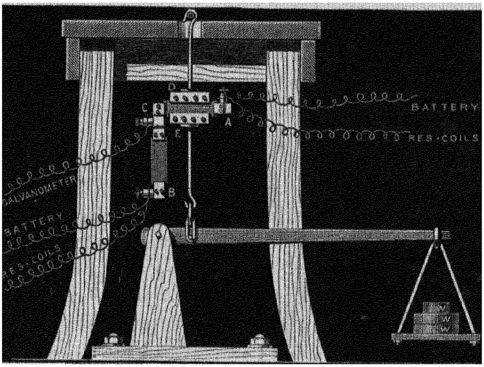
We see that the effect of compression is to diminish both the total resistance and the specific resistance of the specimen of carbon, and is therefore of an opposite nature to that of longitudinal extension.

THE ALTERATION OF ELECTRICAL CONDUCTIVITY PRODUCED BY STRESS APPLIED IN A DIRECTION TRANSVERSE TO THAT OF THE CURRENT.

It has been seen that, in the case of most metals, stress applied in the same direction as that of the current increases, both permanently and temporarily, the resistance, and it seemed probable that stress when applied in a direction perpendicular to that of the current would alter the resistance in the opposite direction. The following arrangements were therefore made with the view of testing this point:—

Strips of metal foil about 10 centims. long, 2 centims. broad, and ·01 centim. thick, were arranged in pairs as in fig. 12. D and E are stout brass clamps into which two hooked iron rods are screwed, and the strip to be examined is clamped into the brass pieces, which are 8 centims. in length, but insulated from them by means of silk. The clamp C connects the metal to be tested with a similar strip, and both strips with one terminal of the galvanometer; whilst the clamps A and B serve, as before, to connect the strips with the resistance coils and battery.

Fig. 12.



The strip to be examined was strained by means of a stout wooden lever, and the table which supported the upper hooked iron rod was surrounded with baize.

Experiment XVI.

A strip of iron foil, 10 centims. in length, 2·1 centims. in width, and of specific gravity 7·65, was subjected to a stress of 120 kilogs., and after two or three applications and removals of this load, it was found that this stress produced a *diminution of resistance* which required the sliding piece to be shifted through 20 divisions of the platino-iridium wire, and that on the removal of the stress the resistance was increased by the same amount. The load was distributed over a sectional area of ·098 square centim., and the diminution of resistance, which would be caused by a stress of 1 grm. per square centimetre, was calculated to be $123·6 \times 10^{-12}$ per unit. Inasmuch, however, as only one-third of the total resistance measured was effected by the stress, it was assumed that if the whole of the piece could have been strained to the same extent the diminution would have been 371×10^{-12}; whereas in the case of iron *wire* subjected to longitudinal traction the alteration of resistance produced by the same stress amounted to 2111×10^{-12}, or nearly six times as much as that caused by transverse traction. Again, if we assume that the value of "YOUNG's modulus" is the same for the foil as for the wire, the total alteration of resistance caused by the transverse traction which would suffice to double the width of the strip would be ·742 per unit. Also taking the ratio of lateral contraction to elongation to be ·281, as was the case with iron (1), the total decrease of resistance due to mere change of form would be 1·562 ; and therefore on the whole there would seem to be an *increase of specific resistance* of ·82, caused by stress sufficing to double the width of the strip.

Experiment XVII.

A strip of tin-foil of nearly the same dimensions as the iron-foil used in the last experiment was tested with a load of 8 kilogs., and a permanent decrease of resistance thereby produced = ·001 per unit; a temporary decrease of ·002 per unit was caused by the application of the load, and it was calculated that the temporary decrease per unit effected by a stress of 1 grm. per square centimetre would amount to 2581×10^{-10} as compared with 1055×10^{-11}, the alteration caused by the same amount of longitudinal stress with tin wire. It would thus seem that, contrary to what takes place with the iron, the alteration produced by transverse traction is much greater (about 23 times as great) than that caused by longitudinal traction; and if we suppose the elasticity of the foil to be even much less than that of the wire, there must be a very appreciable *decrease of specific resistance ;* this latter, if the values of e and σ are the same for the foil and the wire, would be 70·0 per unit, *i.e.*, the metal would have a specific resistance of $\frac{1}{70}$th of its original specific resistance, if stress were applied transversely, sufficing to double the width of the foil. The corresponding alteration produced by the same amount of stress applied longitudinally to a wire of tin is, as we have seen, less than 2 per unit.

Experiment XVIII.

A strip of zinc-foil of nearly the same dimensions as the strips of iron and tin last used was tested with loads varying from 50 to 100 kilogs. A stress of 50 kilogs. produced a permanent decrease of resistance amounting to ·0091 per unit, 70 kilogs. a decrease of ·02 per unit, and with 100 kilogs. the foil was partly pulled away from the clamps; before this, however, the temporary decrease of resistance produced by 50 kilogs. was ascertained to be ·028 per unit. After the accident the foil was securely reclamped, and now three trials with 50 kilogs. gave alteration of temporary resistance amounting to ·010, ·017 and ·010 respectively, the load in the second of the three trials having been allowed to remain for a longer period on the foil than was the case in the other two trials. It was then evident that the time during which the load was allowed to remain on the foil largely influenced the temporary alteration of resistance, and accordingly several experiments were made with a view to verify this fact, which was eventually abundantly established. In one of these experiments a load of 70 kilogs., after having been put on and taken off several times, was allowed to remain on for 10 minutes, when a decrease of ·0428 per unit was measured; after 20 minutes the decrease amounted to ·0557 per unit, part of the decrease produced after the first 10 minutes was permanent, but a considerable part was temporary.

The foil was now tested with loads of 12 and 20 kilogs., the latter causing a decrease of resistance represented by 300 divisions of the platino-iridium wire, and the former 180, that is, the decrease was exactly proportional to the load. From these results it was calculated that a transverse stress of 1 grm. per square centimetre would produce a decrease of resistance amounting to 12384×10^{-11} as against 4406×10^{-12}, the alteration caused by the same longitudinal stress in a zinc wire; the alteration in the former case is about 28 times that in the latter. Again, assuming that the values of e and σ are the same for the foil as for the wire, the decrease of specific resistance which would be caused by transverse strain sufficient to double the width of the strip would be 95 per unit; whilst the alteration for the same amount of longitudinal stress is about 2 per unit. The temporary decrease of resistance of zinc and tin appeared to be so very large, that it was suspected that the silk did not properly insulate the foil from the upper and lower clamps; but this did not seem to be the case, as the resistance was the same before and after clamping; nor could the stress have temporarily impaired the insulation, as if so the same large decrease would have been obtained with iron. It would thus appear that, at any rate for the metals zinc and tin, the effect on the electrical resistance of stress perpendicular in direction to the line of flow of the current is the reverse in every respect of that of stress applied longitudinally. In the case of iron we have seen that though the resistance is on the whole decreased, the mere change of form would more than account for the decrease; and that unless the amount of lengthening which any given load per square centimetre can produce is much less with iron-foil than with iron wire the specific resistance is increased.

One of the most remarkable features of these experiments is the large influence of time on the temporary alteration produced by the heavy loading of zinc and tin. A similar influence, though not to the same extent, was noticed with aluminium, zinc, and tin when great longitudinal stress was employed, and with these metals also for both directions of stress, the departure from proportionality between the load and the alteration produced thereby became very marked, though here again much more so when transverse than when longitudinal stress was applied.

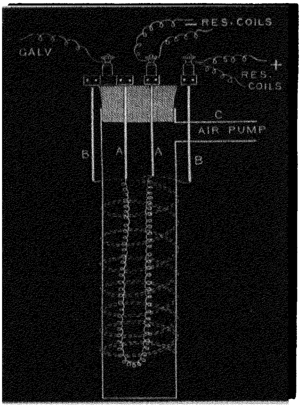
The Alteration of Electrical Conductivity produced by Stress applied equally in all directions.

Unsuccessful attempts and mode of determining the lowering of the melting-point temperature of ice.

So far back as the winter of 1877 I attempted to detect and measure the effect on the electrical conductivity of wires produced by such alterations of fluid pressure as could be obtained by means of the air-pump only, being under the impression at that time that the change of conductivity caused by rise of temperature was due for the most part to mere expansion.

In the first attempt the wire to be tested, a silk-covered copper wire about 12 feet in length and $\frac{1}{40}$th of an inch in diameter, after having been well soaked in melted paraffin wax was coiled in a spiral and placed in a thin, hollow, brass tube (fig. 13), having an inner diameter of 2·8 centims. and a length of 15 centims.

Fig. 13.

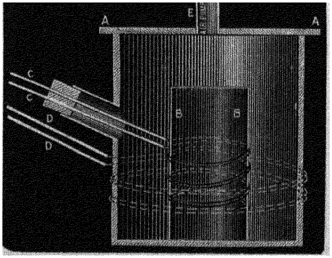


The ends of the wire were soldered to two stout copper wires A A, which passed air-tight through an indiarubber cork and served to connect the wire under examination with the comparison-wire. The latter having its ends soldered to the two stout copper wires B B, was wound double round the outside of the hollow cylinder, and was, together with B and B, secured to the cylinder by string; the comparison-wire had also been soaked in melted paraffin wax, and the four stout wires, A, A, B, B, and their junctions with the other wires, well coated with shellac varnish. The two wires, which as usual were of the same dimensions and substance, were joined up with the other parts of the bridge in the manner already described, and the tube C served to connect the cylinder with an air-pump and barometer-gauge. The whole arrangement having been placed in a box the cylinder was well packed up in sawdust, and after a sufficient time had been allowed to elapse to enable the temperature of the wires to become constant the experiments began.

This plan entirely failed, the effect of the alteration of pressure on the resistance of the wire being completely masked by the change of temperature caused by the rarefaction or condensation of the air even when a considerable time had been allowed to elapse. The condensation and rarefaction of the air would cause a change of resistance the opposite to that looked for as the result of change of pressure; and so slowly did the temperature alter after the first half-hour that I began to suspect that *increase of pressure caused increase of resistance*, and for some days actually tried to measure the *increase* of resistance apparently caused by increase of pressure. Finding, however, that the observations did not agree sufficiently with each other I filled the cylinder with water and immersed it in a large glass vessel, also filled with water, which was stirred from time to time. This plan also failed, the compression of the water causing an effect in the same direction though not to the same extent as the compression of the air had done. It was noticed, however, that after the air was let into the vessel there was *no change of resistance till a second or so afterwards.*

Finally, the following method was adopted:—The vessel containing the wire to be tested consisted (fig. 14) of a brass cylinder closed at one end, and which could be closed at the other by a brass cover A A. The vessel was about 7 inches deep and 6 inches in internal diameter; the open end and the cover which closed it having been carefully ground, so that with the help of a little grease it could be made perfectly air-tight. The wire under examination was wound double on the outside of a very thin brass cylinder B B, and the ends of the wire were soldered to the stout copper wires C, C, which latter passed through an indiarubber cork, fitting air-tight into a tube about 1 inch in diameter, which was soldered into the vessel and made an angle of 45° with the side. The comparison-wire was wound double round the outside of the large cylinder, and the two stout copper wires D, D, to which the

Fig. 14.



ends of the wire were soldered, were connected in the same way as before with C, C and the other parts of the bridge. The whole arrangement of wires and cylinders was placed in a wooden box having holes cut in one of the sides to admit of the connexions C, C and D, D passing through, and this box was in turn placed inside a larger one, so that the space between the two boxes could be filled with sawdust.

The mode of proceeding was as follows:—First, ice broken into small pieces was packed into a layer about 3 inches thick round the larger of the two cylinders, then, after an interval of about half-an-hour, both the large cylinder and B were filled with pounded ice, the cover put on, and over it placed a large sheet of paper containing sawdust. The tube E served to form by means of indiarubber tubing connexion with the air-pump, and after a sufficient time had been allowed to make the balance of the two wires constant the experiments began.

This plan seemed to act admirably. A short time after exhausting the air there seemed to be an increase of resistance, and on letting in the air again a decrease. After each operation with the air-pump the wires were tested at intervals of 10 minutes, until it was certain that the full effect on the resistance of the alteration of the pressure had been produced. Each experiment lasted more than one hour, and it was only possible to make two or three experiments before the ice had to be readjusted round the wires; as evidently after some three hours, though there might be no appreciable melting of the ice, the galvanometer showed that one or other of the wires was not quite at the temperature of the ice. The following are the results obtained:—

Experiment XIX.

Alteration of resistance represented by the number of divisions through which it was necessary to move the sliding-piece in order to restore the balance.	Alteration of pressure in millimetres of mercury.
12 35	700
13 28	721
13·00	730
12·40	718
13 20	710
12·44	707
13 46	707
12·44	707
12·84	708
12·64	708
11·32	708
13·76	708
12·74	707
12 74	707
12·74	707
12 74	710

In these experiments, the divided wire was of German-silver, each division being equal to ·00021105 ohm at the temperature of the room, and since 100 ohms were used at each end of the wire, the alteration of resistance represented by moving the sliding-piece over one division of the wire would equal $\frac{1}{000001055}$ of the whole, the effect of one-tenth of a division was readily perceptible, and the second place of decimals in the above columns were got by taking the mean of several observations at each pressure. From the above results it was calculated that the decrease of resistance per unit produced by an increase of fluid-pressure of 1 grm. per square centimetre would be 27854×10^{-12}.

On showing these results, however, to Professor G. G. STOKES, he suggested that sufficient account had not been taken of the fact that the alteration of pressure might affect the resistance by altering the melting-point temperature of the ice, though, as I thought, the ice would not be sufficiently wet in the inner cylinder, as the temperature of the room at the time was frequently almost at 0° C., and the cylinder well surrounded by ice on the outside to affect the result in this way. Nevertheless, acting on this suggestion, I carefully determined the increase of resistance at 0° C. for a rise of 1° C., in the manner adopted by MATTHIESSEN, and found this to be ·0038587 per unit of resistance. Now, J. THOMSON was led by theoretical considerations* to the conclusion that the melting-point temperature of water would be lowered ·0075° C. by an increase of one atmosphere of pressure, and the matter was put to experimental test by W. THOMSON, who arrived at results agreeing almost exactly with this conclusion. If we assume one atmosphere to be equal to 76 centims. of mercury, or 1034 grms. per square centimetre, the lowering of temperature produced by a pressure of 1 grm. per square centimetre on the melting-point of ice should be ·00000725° C. Now, if the alteration of resistance was entirely due to change of temperature in the melting-point of the ice, my experiments would give a lowering of temperature for a pressure of 1 grm. per square centimetre equal to $\frac{278{\cdot}54 \times 10^{-10}}{{\cdot}003859}$, *i.e.*, to ·00000723° C.

This result then led me to believe that either there was no change of resistance produced by the pressure merely, or, as subsequently proved to be the case, that the alteration of pressure was too small to produce an appreciable effect even with the very delicate galvanometer which I was using. I also remembered that, when water had been employed, as in the previous experiment, as has been before mentioned, there was no appreciable effect *immediately* after the air had been let into the vessel. Moreover, I called to mind that on two occasions when the temperature of the room was almost exactly 0° C. there was no effect, when, as in the last experiments, ice had been used, *even for some* 10 *or* 20 *minutes* after using the pump; but in this case I had attributed the absence of effect to the regelation of the ice over the cylinder round which the wire was wrapped, and had taken the trouble several times to take off the cover and loosen the ice. I have now no doubt that the warmth of the hand in this last act had melted the ice sufficiently to allow the alteration of the melting-point temperature to be felt.

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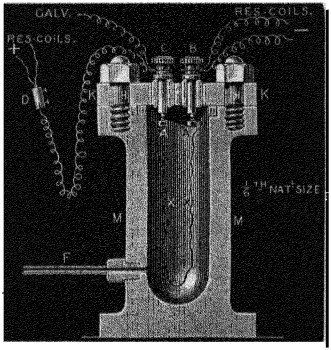
* Trans. Roy. Soc. Edinburgh, Jan., 1849. Cambridge and Dublin Math. Journal, Nov. 1850.

Though these experiments were failures as far as the immediate object in view was concerned, they show what a delicate and accurate thermometer the galvanometer can prove, and that an interesting investigation of the lowering of the melting-point temperature of ice* could be made by using a modification of the above apparatus suitable for pressures of two or three hundred atmospheres, such as will be described presently.

Successful Attempts and Results.

The previous trials having shown that pressures of several atmospheres must be resorted to if one was to hope to measure any alteration of resistance which might be produced by fluid-pressure, the following apparatus (fig. 15) was employed:—

Fig. 15.



M M is a strong vessel made of gun-metal (drawn to scale in fig. 15 one-sixth of the real size), C, B are two binding screws at the ends of two stout brass wires passing water-tight through a cover, K K, and separated from the latter by insulating material. This cover could be removed at will, and the ends of the wire to be tested, X, X, connected by two small binding screws (shown in the figure unscrewed) A, A, at the lower ends of the stout brass wires. After the wire had been well secured to A and A, these latter, the wire itself and those portions of the stout brass wires which projected below the cover, were well coated with paraffin wax. The vessel having been filled with water, the cover with the wire suspended from it was put on and secured by means of three screws and nuts, two of which, H, H, are shown in the figure, the cover having been made to fit water-tight under the highest pressures employed, by means of a gutta-percha collar. The wire X and the comparison-wire were then connected up in the usual way, as shown in the figure,* and the whole well-covered with baize.

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* The effect on the resistance of the wire is so very small as to be almost neglectable, and the correction even for this can be made by data which are given further on. (See p. 77, Table VI.).

The first wire which was tried was a piece of the copper wire used in the last experiments, and a day having been selected at which the temperature was about 4° C., pressure was put upon the wire by means of a force-pump which formed part of a hydraulic press.† The pump communicated with the vessel by means of the tube F, and was capable of supplying a pressure of upwards of 5000 lbs. on the square inch. The pressure was measured by a strong spring pressure-gauge, divided so as to read to a pressure of 100 lbs. on the square inch.

On working the pump so as to increase the pressure there was, after a short time, a very decided decrease of resistance perceptible, and on removing the pressure the resistance returned to almost its former value. Several attempts were made with greater and greater pressures, which were carried up to 4000 lbs. on the square inch, and all gave indications in the same direction; but unfortunately, on raising the pressure to 5000 lbs. on the square inch, the insulating substance round one of the stout brass wires cracked, and allowed the water to come through the cover. As these trials were only intended to be preliminary, no measurements had been taken of the alteration of resistance produced, but there was now no question that in the case of copper wire increase of pressure produced decrease of electrical resistance.

These experiments could not be renewed till the following winter, when the flaw mentioned above having been repaired, some fresh trials were made with iron wire and the following measurements taken:—

Experiment XX.

Pressure in lbs. per square inch.	Number of divisions of the platino-iridium wire through which it was necessary to move the sliding piece in order to restore the balance.
900	20·0
1100	37·7
1300	25·0
Mean 1100	Mean 27·6

---

* In the actual experiments X was secured to the vessel by string, the connexions having been well insulated by wrapping them up in paper.

† This pump was lent to me by the Rev. T. A. Cock, of King's College, who kindly had it put into complete working order before I used it.

These results give an alteration of 1160×10^{-12} per unit in the resistance for a pressure of 1 grm. per square centimetre, and as in the case of copper increase of pressure produces diminution of resistance.

The experiments were again put an end to by a flaw in the insulation, which was made on attempting to carry the pressure to a greater extent, and therefore the final result can only be considered as very roughly correct. Fortunately, however, there is a means of testing approximately the accuracy of this result, as it was found that at the temperature of the room (about 4° C.) the heat caused by the compression of the metal exactly balanced the effect of the pressure on the wire—so exactly, indeed, that even when pressures of 3000 lbs. per square inch were employed, *the instant after the removal of the pressure there was no change whatever in the resistance*, but in a few seconds afterwards the resistance began to decrease, and apparently attained a minimum in about half an hour. About this fact there can be no doubt, as it was repeated several times and with several different pressures. Now the change of temperature produced by an alteration of fluid pressure amounting to 1 grm. per square centimetre is for any substance equal to $\frac{T \times \alpha}{J \times x}$, where $T =$ the absolute temperature, α the coefficient of cubical expansion, J JOULE's equivalent, and x the specific heat referred to unit volume. In this case $T = 278$, $\alpha = \cdot 0000342$, $J = 42{,}400$, $x = \cdot 842$ at the temperature 4° C.

Therefore the rise of temperature caused by an increase of pressure of 1 grm. per square centimetre would be ·000000264° C.; and since the increase of resistance of the wire for 1° C. at 4° C. had by preliminary experiments been determined to be almost exactly ·0047 per unit of resistance, the increase of resistance due to the heat of compression would be $\cdot 000000264 \times \cdot 0047$, or 1241×10^{-12}—a result which agrees fairly with that arrived at by direct experiment.

After making these experiments I was glad to find that CHWOLSON had succeeded in measuring the effect of fluid pressure in the cases of copper, lead, and hard brass,[*] and with these three metals observed that increase of fluid pressure produced decrease of electrical resistance.

In the next table are given the results of CHWOLSON's and my own experiments, in centimetre gramme units.

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[*] Im. Acad. of St. Petersburg Bull., March, 1881, and 'Nature,' June 2, 1881, p. 112.

TABLE VI.

Name of Metal.	Decrease of resistance per unit produced by an increase of fluid pressure of 1 grm. per square centim. = A.	Increase of resistance per unit produced by a longitudinal tension of 1 grm per square centim. = B	Ratio of B . A.	Decrease of resistance per unit attending a fluid pressure sufficing to halve the length of the wire.	Increase of resistance per unit attending longitudinal tension sufficing to double the length of the wire.	Decrease of specific resistance per unit attending a fluid pressure sufficing to halve the length of the wire = C	Increase of specific resistance per unit attending longitudinal tension sufficing to double the length of the wire = D.	Ratio of C : D.
Lead. .	10638×10^{-12}	17310×10^{-12}	1·63	2·440	2·885	3·440	1·613	2·14
Copper .	$1257·0 \times 10^{-12}$	$2310·0 \times 10^{-12}$	1·84	3·470	2·713	4·470	1·005	4·45
Iron . .	$1160·0 \times 10^{-12}$	$2111·1 \times 10^{-12}$	1·82	5·269	4·180	6·269	2·618	2·39
Brass .	$1064·0 \times 10^{-12}$	$2265·5 \times 10^{-12}$	2·13	3·004	2·239	4·004	166	2·41
Means .	.		1·83	.	.	.	.	2·85

Explanation of and remarks on Table VI.

A few words are necessary on the methods of calculating the numbers given in columns five and seven.

If e be the coefficient of longitudinal elasticity, and σ the ratio of lateral contraction to elongation, it can easily be proved that if we subject a wire to a fluid pressure of 1 grm. per square centimetre, the decrease in length per unit thereby produced will be $\frac{1-2\sigma}{e}$. If then A denote the decrease per unit of resistance produced by the pressure, it follows that the decrease attending pressure which would suffice to halve the length of the wire would be $\mathrm{A} \div \frac{1-2\sigma}{e}$, and in this way the numbers in column five have been determined from those in column two; the values of e and σ being those given in Part I., with the exception of the value of σ for brass, which has been taken from MALLOCK's paper,* as I have reason to believe that the values of σ obtained by me for this metal are too large. Again, since the pressure would for such small amounts as are used here cause a decrease of section which would be double the decrease of length, the effect of the pressure in merely altering the dimensions would be to increase the resistance by 1 per unit. In order, therefore, to deduce the values in column seven from those in column five we have only to increase the former by 1.

It will be noticed that the *total* alteration of resistance produced by the fluid pressure is in all cases less than the alteration produced by the same amount of longitudinal stress, the ratio of the latter alteration being to that of the former as 1·83 : 1; but that the alteration of resistance when *the same change of length* is produced by the two kinds of stress is, except in the case of lead, greater when fluid pressure is employed than with longitudinal stress, and that the alteration of *specific resistance* is much greater for the former kind of stress than for the latter, the ratio being about 2·85 : 1.

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* Proc. Royal Society, June, 1879.

The small alterations of resistance which can be produced by fluid pressure as shown in this table prove also how impossible it would have been to detect with such changes of pressure as can be effected by an air-pump any alteration of resistance due directly to compression, and therefore the value of the mode of experimenting already alluded to in determining the amount of lowering of the temperature of the freezing-point of water by pressure : in fact, the change of resistance due to the lowering of the temperature of the melting-point of ice by the pressure would be more than 22 times the change of resistance due to the pressure only.

A brief consideration also suffices to show that the alteration of resistance due to any change of temperature is in all cases very much greater than that which would follow from the same change of volume produced by mechanical stress; this will be seen at once from a glance at the next table.

TABLE VII.

Name of metal.	Coefficient of cubical expansion at 20° C.	Increase of resistance per unit caused by a rise of 1° C. at 20° C.	Rise of temperature necessary to double the volume = A.	Increase of resistance per unit caused by rise of temperature A = B.	Increase of resistance per unit if the wire could have its volume doubled by mechanical stress = C.	Ratio of B : C.
			° C.			
Lead . .	$\cdot 8223 \times 10^{-4}$	00375	12,160	45 6	0·962	47 4
Copper .	4554×10^{-4}	·00380	21,960	83 5	0·904	92·4
Iron . .	$\cdot 3420 \times 10^{-4}$	·00470	29,250	137 5	1 393	98 7
Brass . .	$\cdot 5450 \times 10^{-4}$	·00122	18,350	22 4	0 746	30 3

We see from this last table that the alteration of resistance due to any change of temperature is in the case of the four metals, lead, copper, iron, and brass, from 30 to nearly 100 times as great as that which would follow from the same change of volume produced by mechanical agency; and it would appear, again, therefore, that the increase of resistance caused by rise of temperature is principally due to other causes than mere expansion.

PERMANENT ALTERATION OF RESISTANCE PRODUCED BY LONGITUDINAL TRACTION.

Limit of elasticity.

Experiments on the permanent alteration of resistance produced by traction possess considerable advantages in determining the limit of elasticity over the methods usually adopted. In the first place, it is possible to detect much more minute elongations, even though very small lengths of the wire be employed, than would be the case if 100 feet of the wire could be tested in the ordinary way with the cathetometer, and in the second, all errors arising from the wires not being perfectly straight at the commencement of the experiments are entirely avoided.

The limit of elasticity raised by previous loading.

ROBERT THALÈN[*] with, no doubt, others has shown that in the cases of iron and steel the limit of elasticity can be raised by previous loading.

This was found to be so for all the metals examined, and the next experiment furnishes a fair example of the kind.

Experiment XXI.

An annealed copper wire was very carefully loaded and unloaded by allowing water to flow slowly into and out of a pail attached to it, and the permanent increase of resistance determined after each unloading. The water entered the pail from a vessel containing 5814 grms. of the liquid, and this quantity took five minutes to pass into or out of the pail.

Number of trial.	Number of measures each = 5814 grms.	Number of divisions through which it was necessary to move the sliding-piece in order to restore the balance after each unloading.
1	1	14
2	1	1
3	2	31
4	1	0
5	2	2
6	3	54
7	1	0
8	2	2
9	3	9
10	4	80
11	1	0
12	2	0
13	3	0
14	4	7
15	5	122
16	1	0
17	2	0
18	3	0
19	4	0
20	5	25
21	6	235
22	1	0
23	2	0
24	3	0
25	4	0
26	5	8
27	6	29
28	7	1084

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[*] Pogg. Ann., April, 1865, and Phil. Mag., December, 1865.

Critical points.

THALÈN has also proved* that if a curve be drawn having for its ordinates and corresponding abscissæ lines proportional respectively to the permanent extensions and the load producing them, there is a point of maximum curvature, where the increase of length becomes suddenly large compared with the load, and suggests this point, which appears to be nearly at the same part of the curve for different specimens of iron and steel, as a suitable substitute for the so-called "limit of elasticity."

My own investigations have shown that there exist in every metal *two* such points, which I have called critical points, and these are evidently very closely related to the moduli of elasticity. The first of these critical points is one that must of necessity have escaped the notice of most observers of the elongation of wires in the usual manner, as the load required to straighten the wire sufficiently for observations made in this way is beyond this point. Now in the present investigations it is not requisite to straighten the wire, and as the increase of resistance proves to be almost exactly proportional to the permanent elongation, it is evident that the curves showing the relations between permanent extension and load will be similar to those connecting permanent increase of resistance and load.

Experiment XXI. furnishes a good example of the first critical point. It will be seen that the first critical point occurs at the seventh measure, this producing a permanent increase of resistance the ratio of which to that produced by the previous measure is appreciably greater than any of the other similar ratios.

The following experiment is a sample of a series undertaken with a view to ascertain the position of the two critical points for the different metals.

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* Phil. Mag., December, 1865.

Experiment XXII.

A well-annealed silver wire was suspended for trial with the scale-pan attached, and loads increasing by 1 kilog. at a time were put on for three minutes and then removed, when the permanent increase of resistance was determined for each load.

Load in kilogs		Increase of resistance in divisions of platinum silver wire.
	1st	40
	2nd	40
	3rd	45
	4th	60
	5th	75
	6th	80
First critical point .	7th	142
	8th	186
	9th	306
Second critical point	10th	1436
	11th	2706

Each division of the platinum-silver represents in this case an increase of resistance of only $\frac{1}{1240}$ per cent., and as the percentage of permanent elongation is half of this fraction, because the wire is decreased in section very nearly in the same proportion as it is increased in length, it follows that the total elongation produced by all the loads up to the first critical point inclusive, does not exceed ·2 per cent.

The case of iron.

The behaviour of iron under longitudinal traction is very remarkable; this metal, if the load be applied in small quantities at a time, is seen at certain points to become perfectly rigid, so that the further application of stress does not produce any further permanent elongation until the load has reached a certain value, when elongation once more commences.* Further, iron possesses three critical points at least, and may be found to have more;† these points are well shown in Experiment XXIII.

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* This was first noticed by J. T. BOTTOMLEY (Proc. Roy. Soc., No. 197, 1879), who kindly lent me some of the same kind of wire as that used by himself.

† Perhaps other metals may have more than two critical points, which might be discovered by loading the wires more gradually.

Experiment XXIII.

ANNEALED iron wire.

Load in kilogs.		Permanent increase of resistance represented by divisions of platinum-silver wire.
	1st	5
	2nd	8
	3rd	17
	4th	21
	5th	29
	6th	33
	7th	42
First critical point	8th	120
	8½	118
	9th	40
	9½	25
	10th	30
	10½	35
	11th	0
	11½	15
	12th	83
	12½	177
	13th	223
	13½	389
Second critical point	14th	2356
	14½	2028
	15th	1421
	15½	2052
	16th	36
	16½	24
	17th	304
Third critical point	17½	1603
	18th	Broke

It will be seen from the above experiment that there is a critical point at the 8th kilog., another at the 14th kilog., and a third at the 17th kilog.

The two critical points were determined in all cases by finding the two points where the ratio of the increases of resistance produced by consecutive equal loads is greatest; the next experiment furnishes an example of the method adopted.

Experiment XXIV.

ANNEALED German-silver (2).

Load in kilogs.	Increase of resistance in divisions of the platinum-iridium wire.	Ratio of consecutive increases of resistance.
1st 2	21	
2nd 2	51	2·43*
3rd 2	59	1·15
4th 2	65	1·10
5th 2	90	1·38 1st critical point
6th 2	117	1·30
7th 2	143	1·22
8th 2	326	2·28
9th 2	1083	3·32 2nd critical point
10th 2	1156	1·07

The weight of the scale-pan, pulley, &c., was 2·5 kilogs., or 1·25 kilogs. on each part of the wire, therefore the total load at the first critical point is equal to 11·25 kilogs., and that at the second 19·25 kilogs. Now the section of the wire at the two points is ·00883 and ·00876 per square centimetre; hence the load at the first critical point is 1274 and at the second 2197 kilogs. per square centimetre.

In Table VIII. will be found the loads at the two critical points and their relation to "YOUNG'S modulus."

TABLE VIII.

Name of metal.	Value of e in grms per square centim.	Load at the first critical point in grms. per square centim. $= \alpha$.	Load at the second critical point in grms. per square centim. $= \beta$.	Ratio e α.	Ratio e β.
Iron (1)	1981×10^6	$2{\cdot}070 \times 10^6$	$3{\cdot}009 \times 10^6$	$9{\cdot}6 \times 10^2$	$6{\cdot}6 \times 10^2$
Nickel (1) .	2175 ,,	2·200 ,,	3·100 ,,	9·9 ,,	7·0 ,,
Platinum (1)	1490 ,,	1·400 ,,	1·520 ,,	10·6 ,,	9·8 ,,
German-silver (2).	1291 ,,	1·274 ,,	2·197 ,,	10·1 ,,	6·0 ,,
Copper (2)	1218 ,,	1·151 ,,		10·7 ,,	
Copper (1) . . .	1160 ,,	1·115 ,,	1·702 ,,	10·4 ,,	6·8 ,,
Copper (3)	1143 ,,	1·230 ,,	1·870 ,,	9·3 ,,	6·1 ,,
Platinum-silver (1) . .	1051 ,,	1·081 ,,	1·741 ,,	9·8 ,,	6·0 ,,
Silver (1)	742 ,,	·819 ,,	1·250 ,,	9·1 ,,	6·0 ,,
Aluminium (1).	673 ,,	·730 ,,	1·120 ,,	9·2 ,,	6·0 ,,
Mean for all the different metals	. .	.	.	$9{\cdot}8 \times 10^2$	$6{\cdot}7 \times 10^2$

---

* In this, as in several other experiments, the first of the ratios of consecutive increases is greater than several of those which follow, but this is probably due to the fact that the weight of the scale-pan had been acting on the wire for some hours previous to the period of actual testing.

Remarks on Table VIII.

It is evident from the last table that the loads both at the first and second critical points, in the case of each substance, bear a constant ratio to the corresponding value of "YOUNG's modulus," there being quite as much difference between the several ratios $\frac{c}{\alpha}$ and $\frac{c}{\beta}$ for the three specimens of copper as for the various metals. The time during which the load was allowed to act was in all cases the same, namely, three minutes,[*] and the wire was relieved of all weight except that of the scale-pan before determining its alteration of resistance.

Experiment XXI. having shown that the position of the critical points must be altered by the process of wire-drawing, it is necessary that very great care should be taken in annealing the wire[†] if we wish to determine the true position of these points for any substance whose particles are to be free from mutual strain previous to beginning the experiments. The theoretically correct definition of the "limit of elasticity" would be the highest load per unit of surface which a wire can bear without undergoing the slightest permanent elongation. It is clear, however, that we cannot, even with the utmost care, obtain a substance which will have its particles in a *perfectly homogeneous* condition, and it will of necessity happen that *some* of the particles are on the point of passing the elastic limit or, at any rate, that some are much nearer this limit than others, before any external stress has been applied. Consequently we can never hope to obtain the true value of the elastic limit by merely endeavouring to observe the first trace of a permanent elongation, our power to do so depending upon the delicacy of our instruments; but we can accomplish our object in all probability by determining the point at which the ratio of the elongation and the stress producing it reaches its first maximum. I would venture, therefore, to suggest, that the *first*[‡] of the two so-called critical points be taken to represent the true "limit of elasticity" of a well-annealed substance, and to agree with THALÈN that neither the method of measuring the limit of elasticity by the greatest load which will produce sensible permanent elongation, nor the purely arbitrary one adopted by WERTHEIM and others of fixing an elongation of ·0005 of the unit length as corresponding to this limit, is desirable.

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[*] Except in the case of copper (3) where five minutes was allowed.

[†] It is certainly not sufficient to heat a wire to redness *for a few minutes*, and then allow it to cool slowly; the high temperature must be maintained in some cases for a considerable time.

[‡] The *second* of the critical points is evidently the same as that suggested by THALÈN as being suitable for measuring the limit of elasticity (Phil. Mag., December, 1865), but I think that the second point is rather the precursor of breakage than of the passage of the elastic limit.

Critical points of tin and lead.

The positions of the critical points of these metals have not been satisfactorily determined though several trials were made for that purpose, and in some of these great care was taken to load by very small amounts at a time and to allow each weight to remain on the wire for a considerable period; in the following experiment each load was left on the wire for 48 hours :—

Experiment XXV.

ANNEALED lead wire.

Load in tenths of a kilog.	Increase of resistance in divisions of the platinum iridium wire.	Ratio of consecutive increases of resistance.
1st tenth . .	7 0	
2nd . . .	16·5	2·4
3rd	38·0	2·3
4th . . .	49·0	1 3
5th . . .	66·0	1·4
6th	103 0	1·6
7th . .	207·0	2·0
8th	540 0	2·6
9th	wire broke.	

If we regard the second load as the load at the first critical point, and the eighth as the load at the second critical point, we obtain for the loads at the two critical points respectively the values 48 kilogs. per square centimetre* and 74 kilogs. per square centimetre; in this case $\frac{e}{\alpha}$ would $=35\times 10^2$, and $\frac{e}{\beta}=22{\cdot}6\times 10^2$, both these numbers being much larger than the values of $\frac{e}{\alpha}$ and $\frac{e}{\beta}$ for the metals in Table VIII.

With tin, also, the loads at the first and second critical points, as far as the latter could be determined,† seemed much less in proportion to the value of e than was the case with most of the other metals.

In Table IX. the permanent increase per cent. of resistance produced by longitudinal traction is shown for each metal by a series of curves; and these curves will also show very fairly the permanent increase per cent. of length produced, since the former is very nearly equal to twice the latter.

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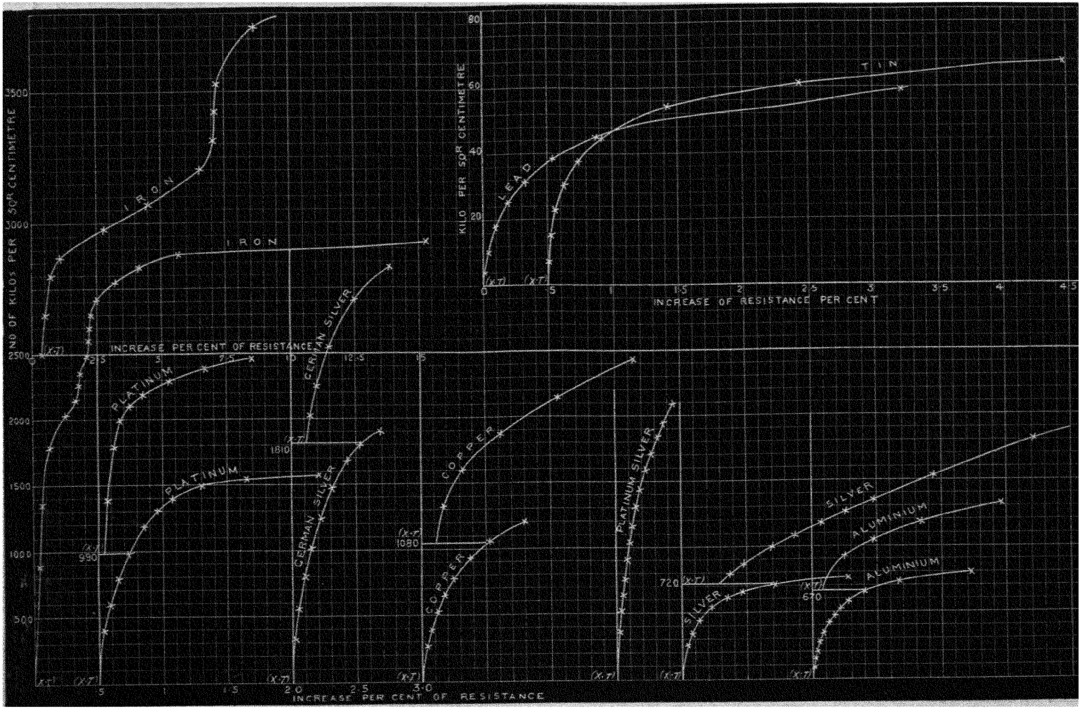
[*] 30 kilogs. per square centimetre added for the permanent load, consisting of pulley, &c.

[†] I am inclined, however, to attach very little value to the observations made on the critical points of lead and tin; very probably better results might be obtained by using wires of much greater section than those here employed.

TABLE IX.



NO OF KILOS PER SQR CENTIMETRE
3500
3000
2500
2000
1500
1000
500
IRON
IRON
PLATINUM
PLATINUM
GERMAN SILVER
GERMAN SILVER
COPPER
COPPER
PLATINUM SILVER
SILVER
SILVER
ALUMINIUM
ALUMINIUM
(X.T.)
990
1810
1080
720
670
INCREASE PER CENT OF RESISTANCE
2.5
5
7.5
10
12.5
15
INCREASE PER CENT OF RESISTANCE
.5
1
1.5
2.0
2.5
3.0
KILO PER SQR CENTIMETRE
80
60
40
20
LEAD
TIN
INCREASE OF RESISTANCE PER CENT
0
.5
1
1.5
2
2.5
3
3.5
4
4.5

Explanation of and remarks on Table IX.

The curves are in most cases divided into two parts, the lower part representing the increase of resistance per cent. up to and a little beyond the first critical point, and the upper one the increase up to and beyond the second critical point. The abscissæ represent the increase of resistance per cent. for the lower curves on a scale of 40 millims. to unit increase per cent., and for the upper ones on a scale of 80 millims. to unit increase per cent.

The ordinates represent the load per square centimetre in kilogrammes, and are on a scale of 40 millims. to each 1000 kilogs. for the lower curves and of 80 millims. to each 1000 kilogs. for the upper ones.

The upper curves start from the horizontal lines drawn through their lower extremities, and the starting points on these lines are set off at distances representing on the scale for the upper curves the increase of resistance already attained: also opposite each horizontal line is placed the number of kilogrammes per square centimetre already put on to the wire. In all cases the origin of coordinates is marked X, T.

Thus, for example, the upper extremity of the lower copper curve has an abscissa $=32$ millims. and an ordinate of 48 millims, therefore the load at this point is $48\times\frac{1000}{40}$ or 2400 kilogs. per square centimetre; and the increase of resistance is $32\times\frac{1}{40}$ or 8 per cent. Similarly the upper extremity of the upper copper curve has an abscissa of 65·6 millims. and an ordinate reckoned from the horizontal line passing through the lower extremity of the curve of 56·8 millims., therefore the total load is $56{\cdot}8\times\frac{1000}{80}+1080$ or 1790 kilogs. per square centimetre, and the increase of resistance $65{\cdot}6\times\frac{1}{8}$ or 8·2 per cent.

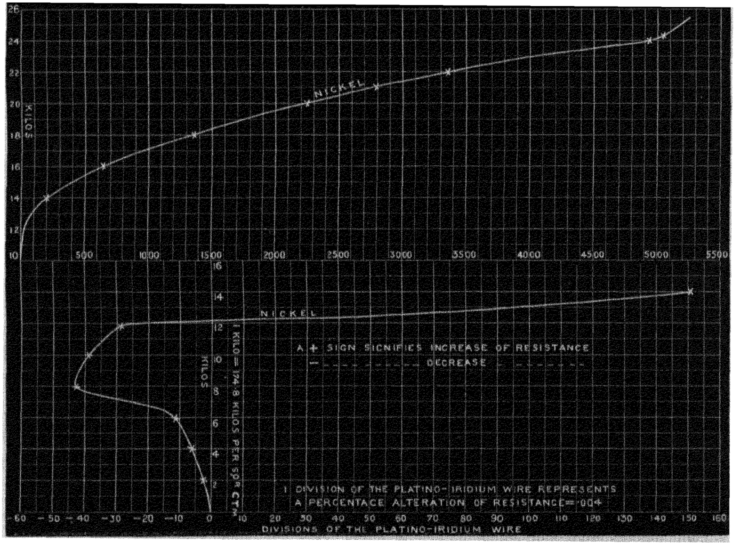
In order to include lead and tin in the same table it was found necessary to reckon the load on a scale of 2 millims. to 1 kilog. per square centimetre, and the increase of resistance on the same scale as that used for the lower curves of the other metals. Moreover, to the load registered in the table there should be added for these metals 30 kilogs. per square centimetre, this representing the permanent load on the wires.

It will be observed that there is in most cases a considerable resemblance between the forms of the upper and lower curves, and that in the case of iron the curve near the two critical points becomes very nearly a vertical straight line.

The critical points of nickel.

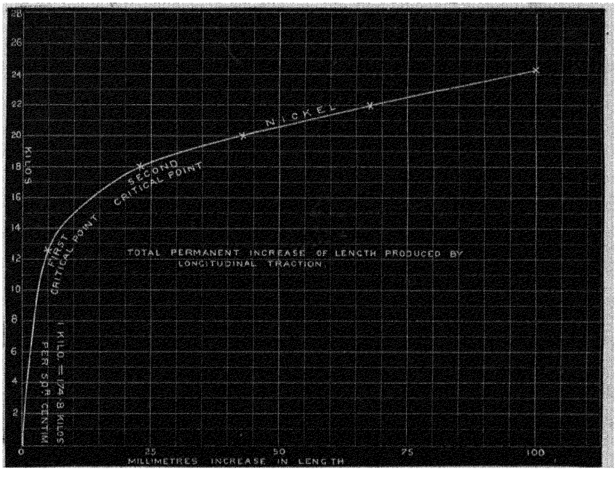
We have seen that the electrical resistance of nickel is altered in a peculiar manner by temporary longitudinal traction, and we might expect, therefore, that the effect of permanent extension would perhaps be different in the case of this metal, both in character and extent, from what it is with other substances. This is found to be so, and Table X. shows that the total permanent alteration of resistance produced by permanent extension is in the first instance of the nature of a decrease, but that, after a certain load has been reached, the resistance begins, as in other metals, to increase. The curves in Table X. are drawn with their abscissæ to represent the alteration of resistance—for the lower one on a scale of 1 division of the platino-iridium wire to the millimetre, and for the upper one on a scale of 25 divisions of the platino-iridium wire to the millimetre.

TABLE X.—Curve showing the total permanent alteration of electrical resistance produced by longitudinal traction from 10 to 24½ kilogs., and curve showing the permanent total alteration of electrical resistance produced by longitudinal traction from 0 to 14 kilogs.



In both curves the ordinates represent the number of kilogrammes on the wire to a scale of ·2 kilog. to the millimetre. The lower curve which represents the permanent alteration of resistance produced by loads up to 14 kilogs., shows that the resistance decreases to an extent which is nearly proportional to the load up to 6 kilogs.; here a sudden leap is made of rapid decrease of resistance, but on reaching the next load, 8 kilogs., the curve turns, and just at the 12th kilog. there occurs a sudden and rapid increase of resistance. The upper curve which starts from the 10th kilog. shows that the rapid increase begun at the 12th kilog. is continued to an extent which is nearly proportional to the load up to the breaking point of the wire. The nature of these curves would seem to show critical points at the 6th, 8th, and 12th kilogs., but as the effect on the *specific resistance* of this metal was found to be very great in comparison with that of the other substances, it was evident that the critical points as understood to mean points of sudden increase of *length* compared with the load producing it could not safely be deduced from the curve showing the total alteration of resistance. The critical points have therefore been deduced from direct observations of the permanent increase of length produced by each load, and the results of these observations are shown in the curve of Table XI. This curve has its abscissæ representing the permanent increase of length on a scale of ·625 millim. of increase of length to the millimetre and its ordinates the load in kilogrammes on a scale of ·2 kilog. to the millimetre. From the observations of increase of length the first critical point was determined to be at the load 2200 kilogs. per square centimetre and the second at the load 3100 kilogs. per square centimetre.

TABLE XI.—Total permanent increase of length produced by longitudinal traction.



THE INFLUENCE OF TIME ON THE AMOUNT OF PERMANENT INCREASE OF RESISTANCE OR OF LENGTH WHICH CAN BE PRODUCED BY LONGITUDINAL STRESS.

It is well known to all who have made investigations in the subject of elasticity, that the permanent increase of length which can be produced by a given amount of longitudinal stress depends largely upon the time during which the stress is allowed to act, and as it was expected that experiments on the permanent increase of resistance would throw some light on this influence of time, several trials were made with a view of ascertaining whether a wire will, when under the action of considerable stress, show greater and greater increase of length until it breaks, or whether the time-curve of its increase of length will be asymptotic.

Experiment XXVI.

A piece of copper (3) having been loaded for some time with six measures of water was still further loaded with one measure, this being allowed to enter very slowly into the pail attached to the wire; the current from a DANIELL'S element was kept flowing through the "bridge" circuit, and the alteration of resistance observed by noting the position of the light on the scale.

Time in minutes after the completion of the loading.	Increase of resistance* for each minute in terms of the deflection of the light in scaled divisions.	Difference of increase for consecutive minutes.
1	45	. .
2	31	14
3	25	6
4	24	1
5	18	6
6	16	2
7	16	0
8	15	1
9	17	−2
10	10	7
11	13	−3
12	14	−1
13	13	1
14	7	6
15	5	2
16	8	−3
17	10	−2
18	8	2
19	5	3
20	7	−2

From the last experiment we learn that, though the wire had been so carefully loaded and the room free from vibrations of any sort, the increase of resistance did not, *shortly after loading*, take place smoothly, but the difference of increase of resistance for consecutive minutes became alternately greater and less.

In the next experiment the increase of resistance is measured by the number of divisions of the platino-iridium wire through which it was necessary to move the sliding-piece in order to restore the balance, and the times recorded are those taken to produce the increase of resistance.

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* Each scale-division here represents an alteration of resistance equal to about ten-millionths of the whole.

Experiment XXVII.

The same wire as in the last experiment was allowed to run down for a few minutes longer, and then the following observations were made :—

Increase of resistance in divisions of the platino-iridium wire.		Time in minutes.	Ratios of consecutive times.
1st . .	40	13·0	
2nd	40	19 0	1 462
3rd . .	40	24 0	1 262
4th .	40	30·0	1 250
5th . .	40	41·8	1·394
. . .	.	. . . .	. .
. .		.	. . .
11th . .	83 40	415	
Mean value of ratios = 1 342.			

It is to be noticed that here the increase of resistance takes place more regularly; and though the ratios of consecutive times are not very constant, it would seem that the times form a geometrical progression, since the time for the last increase of resistance, as calculated from the formula $t=13\times 1{\cdot}342^{11\,83}$, where t denotes the time, is 422 minutes—a number agreeing sufficiently well with the observed time.

The wire was afterwards loaded more and more, and the values of the ratios of consecutive times, as determined in the same manner as in the last experiment, are recorded in Table XII.

TABLE XII.

Resistance of the wire in terms of the comparison-wire.	Total load on the wire in measures of water, each measure = 5·825 kilogs	Calculated load in kilogs. per square centim.	Percentage value of the increments of resistances represented by 40 divisions of the platino-iridium wire.	Ratios of the consecutive times required by the loads to produce equal increments of resistance.											Mean values of ratios.	Mean values of ratios calculated for a percentage increment of resistance = ·1.
1·056	9 10	1448	·0314	1·24	2·00	1·21	1·33	1·29	1 17	1·38	1·50	1 27	1·24	1 22	1·35	2·572
1·080	9·75	1570	·0311	1 38	1 88	1·32	1·35	1·08	1·21	1·18	1·30	1·20	1·24	..	1·26	2·097
1·120	10·75	1759	·0305	1·10	1·21	1·07	1·23	1·14	1·17	1·14	1·25	1·18	1·13	1·25	1·17	1·721
1·177	11·75	1971	·0298	1 12	1·15	1·13	1·16	1 08	1·12	..	..	..	..	..	1·13	1·492

We learn from Table XII. that the ratios of consecutive times become more and more constant as the load on the wire is made larger and larger, and therefore that the velocities of increase of resistance for small equal increments of resistance form a geometrical progression.

In order to make the mean values of the common ratios comparable with each other, the common ratios calculated for equal increments of resistance of ·1 per cent. are recorded in the last column; the calculation was effected by raising the observed ratio in the last column but one to the power obtained by dividing ·1 by the percentage values of the equal increments of resistance given in the fourth column. For example, with a load of 1448 kilogs. per square centimetre, the common ratio of the geometrical progression found by the times taken for the load to increase the resistance by successive percentage amounts of ·0314 was found to be 1·35, therefore the ratio which would have ensued, if the times taken to increase through ·1 per cent. had been observed, would have been $1{\cdot}35^{\frac{1}{0314}}$ or 2·572.

Again, the loads in kilogrammes per square centimetre given in the third column are calculated from the resistances given in the first column in the following manner: let S_1 and S_2 be the sections of the wire before and after stretching, and let the corresponding resistances be R_1, R_2, and the lengths l_1 and l_2; then, $\frac{R_2}{R_1}=\frac{l_2}{l_1}\times\frac{S_1}{S_2}$, provided that there be no change in the specific resistance of the metal; also $l_1\times S_1=l_2\times S_2$, or $\frac{l_2}{l_1}=\frac{S_1}{S_2}$, provided that the stretching does not alter the density. But we shall see that neither the density nor the specific resistance is altered by stretching to any extent sufficient to introduce any appreciable error, therefore we have within a sufficiently close approximation $\frac{R_2}{R_1}=\left(\frac{S_1}{S_2}\right)^2$, or $S_2=S_1\times\sqrt{\frac{R_1}{R_2}}$. Thus the section S_1 having been determined, it is easy to ascertain the section after any amount of stretching. For example, the section of the wire last used was before stretching ·0183 square centim., and the resistances before and after the stretching were 1·056 and 1·080 respectively, therefore the section of the wire after stretching would be nearly $\cdot0183\times\sqrt{\frac{1\cdot056}{1\cdot080}}$, and the actual load on the wire being 9·75 measures of water or $9{\cdot}75\times5{\cdot}825$ kilogs., the load per square centimetre would be $\frac{9\cdot75\times5\cdot825}{\cdot0183}\times\sqrt{\frac{1\cdot080}{1\cdot056}}$ kilogs. $=1570$ kilogs. In all cases the resistances recorded in the first column are the means between the resistances observed at the commencement and end of the times during which the velocities of increase were noted, and the wire was allowed to run down for some time (about 20 minutes) before the observations of the velocities of increase commenced.

It is, moreover, evident from the last column in the table that the common ratio of the geometrical progression becomes less and less as the load becomes larger and larger, and we might expect to find some relation between the decrease of velocity of increase of resistance and the difference between the breaking-load and the load actually on the wire. A further examination of the results given in Table XII. shows that a relation does exist of an extremely simple nature; in fact, if we denote the breaking-load and the actual load on the wire by P and p respectively, the decrease per unit of the velocity of increase of resistance is inversely proportional to $P-p$. The value of P was carefully determined by loading pieces of the wire by moderate amounts at a time, and allowing each load to remain on the wire some ten minutes before each further addition, and measuring the diameter *close to the point of breakage* by means of a wire-gauge graduated to ·01 millim. The mean of several trials gave a value for P of 2625 kilogs. per square centimetre.

Now if t_1, t_2, t_3, &c., are the times taken to increase the resistance by successive equal and small amounts, the ratios $t_2 : t_1$, $t_3 : t_2$, &c., have been proved to be constant for the same value of p: let this constant be denoted by r; then the decrease per unit of the velocity of increase of resistance being $=\frac{1}{t_1}-\frac{1}{t_2}\div\frac{1}{t_1}$ will therefore $=\frac{r-1}{r}$, and accordingly if the above-mentioned relation holds good, $\frac{r-1}{r}\div(P-p)$ should be a constant for different values of p. The last column of the table gives the values of r for the loads given in the third column, and the values of $\frac{r-1}{r}\div(P-p)$ are for the four loads there recorded, ·000519, ·000496, ·000484, and ·000504, with a mean value of ·000501.

The question next arises, will the ratio $\frac{r-1}{r\times(P-p)}$ be the same for other metals as for copper? for if so we can calculate the breaking-load by merely loading the wire beyond the second critical point and observing the value of r; then, since $\frac{r-1}{r\times(P-p)}=\cdot000501$,

$$P=p+\frac{r-1}{r\times\cdot000501}.$$

With a view of ascertaining whether the breaking-load could be thus calculated, a series of experiments with different metals was begun, the mode of operating being similar to that just described in the case of copper, and the results are shown in the next table. In the same table are also given the ratios of the different moduli of longitudinal elasticity to the corresponding breaking-load.

TABLE XIII.

Metal.	Percentage value of the equal increments of resistance.	Actual load on the wire in kilogs per square centim. $= p$.	Ratio of consecutive time calculated for an increase of ·1 per cent. of resistance $= r$	Calculated breaking load $P = p + \frac{r-1}{r \times \cdot 000501}$	Observed breaking-load in kilogs. per square centim.	Mean values of breaking-loads from observation and calculation.	Ratio of "YOUNG'S modulus" to breaking-load.
Copper (3) . .	·0314	1448	2·572	2666	2625	2625	4 35 × 10^2
	·0311	1570	2·097	2612			
	0305	1759	1·721	2593	. .		
	·0298	1971	1·492	2628			
Iron (1) . . .	·2270	3311	1 081	3826	3816	3825	5·17 „
					3785		
					3867		
Zinc (2) . . .	9620	1418	1·014	1446	1468	1460	5 25 „
					1528		
					1426		
Platinum (1) .	·1524	1737	1·131	1968	2061	2021	7 20 „
	·1487	1836	1·064	1956	2080		
	·1482	1946	1 029	2004	2099		
Silver (1) .	·0609	1076	2 531	2281	2090	2185	3·44 „
	0601	1192	2 015	2195	2272		
	·0582	1440	1 498	2099	2170		
Aluminium (1)	·0552	864	1 144	1295	1277	1293	5 19 „
					1304		
					1288		
							Mean 5·10 × 10^2

Observations on Table XIII.

We learn from the above table how very closely in all cases the calculated breaking-load agrees with the observed breaking-load, though the values of p and of the percentages of increase of resistance for which the times were observed varied considerably. What differences do exist are no greater than those between the values of the observed breaking load for different pieces of the same wire.

Since, also, the permanent increase of resistance produced when a wire is running down under the influence of a load is for small amounts nearly double the increase of length, we may determine the breaking load by observing the times taken by the load to produce successive equal increments of length amounting to ·05 per cent., and use the same formula as above.*

In estimating the breaking-load as has been before mentioned, the diameter of the wire was gauged in the immediate neighbourhood of the breaking-point; and this is necessary if we wish to determine the true breaking-load in kilogrammes per square centimetre, inasmuch as, however uniform in diameter, and however carefully annealed the wire may have been before stretching, the latter action is sure to diminish the diameter at some parts of the wire more than at others, and eventually the wire breaks at that part at which the greatest contraction has taken place.

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* It should be noticed that the value of r as used in the formula is greater than unity, and is obtained by dividing *the succeeding by the preceding time* of accomplishing any two consecutive increments of resistance or of length.

If a wire could be obtained of perfect uniformity of diameter and substance, the contraction would gradually increase from each end to the centre where it would be greatest, and at this point the wire would break. It is interesting to watch this gradual increase of contraction from the two ends to the middle in the case of a test-bar of ductile iron or steel, and through the kindness of Sir JOSEPH WHITWORTH I was able to make the following experiment on such a bar of fluid-pressed steel, the specimen having been selected as suitable for this purpose in consequence of its great ductility.

The bar had a total length of 6 inches before stretching, but a screw was formed, 2 inches in length (see fig. 16), at each end for the purpose of securing the bar in the framework of the hydraulic press employed to stretch it. The diameter of the bar between the two screws was before testing ·7979 inch, and after each stretching the bar was removed from the press and the length and diameter re-determined: the results obtained are recorded in the next table.

Fig. 16

TABLE XIV.

Stress in tons per square inch.	Total permanent increase of length measured after the removal of the stress.	Total permanent contraction of diameter at the centre.	Ratio of permanent contraction of diameter to permanent elongation.
20	·1060	·025	·236
22	1560	·030	192
24	·2380	050	211
25	·3010	053	·176
26	·5175	098	190
26 Second time	·7850	·228	·294
23½ Broke	·8900	323	363

It will be noticed that the ratio of permanent contraction of diameter to the permanent elongation is as sensibly constant as could have been expected from the way in which the diameter had to be measured, namely, by calipers,* until the breaking-load is nearly reached, when the contraction begins very suddenly to increase.

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* WHITWORTH's measuring-machine would have been employed for this purpose as it was for measuring the increase of length, had not the curving of the bar produced by the stretching rendered such an instrument unsuitable.

Fig. 16 is from a photograph of a similar specimen, broken after stretching, and shows the gradual contraction from the ends to the centre and the sudden increase of contraction at this point. In measuring the diameter of the broken wires the gauge was placed close to the place of this sudden contraction.

It appears also from Table XIII. that there is a certain amount of relationship between the breaking-load and the modulus of longitudinal elasticity; platinum, however, having too small a breaking-load and silver too high in comparison with the modulus of elasticity when contrasted with the other metals.

It should be remembered that in the case of the former metal the loads at the critical points were less in proportion to the elasticity than was the case with the other metals, and if we regard the breaking-point as a third critical point, it would appear that these three critical points are in the case of well annealed wires related to each other roughly in the ratios of 1 : 1½ : 2 or of 2 : 3 : 4.

THE PERMANENT ALTERATION OF SPECIFIC RESISTANCE CAUSED BY STRESS.

Traction.

We have seen that the alteration of specific resistance which can be temporarily produced by longitudinal traction is very small, but then the temporary lengthening is small also. Now we can with wires which have been well annealed produce, in most cases, a far more considerable permanent increase of length, and it seemed desirable to ascertain whether there would be a correspondingly large change in the specific resistance of the substance.

Three different modes of experimenting were tried:—in the first, the wire to be tested and the comparison-wire were clamped into the blocks already described, and were then stretched at full length on the floor and side by side; the short block, into which one end of the wire to be stretched was clamped, was placed behind two stout screws, which were screwed into the floor to about one half of their length, and about one inch apart; the wire passed between these screws, and the other end of it which was clamped into the longer brass block was pulled by hand until it was quite straight· a mark on the block which traversed a wooden scale divided into millimetres, and secured to the floor by screws in a position parallel to the length of the wire, serving to measure the length of the wire. It was ascertained that after a little practice the readings could be depended upon to within at least 1 millim. or about $\frac{1}{2400}$ of the whole length. After thus measuring the length, the wires were placed in the air-chamber already described, and after the usual precautions, the ratio of their resistances tested; a weight was now placed on the pulley for three minutes and then removed, when the wire was permitted to remain free from stress for 10 minutes, after which time a new determination of the ratio of the resistances of the wires was made.

The wires were now removed from the air-chamber and the one which had been stretched was remeasured. The same processes were repeated after greater and greater loads until finally the wire was broken. With some of the wires the specific gravity was determined after each stretching, but as the alteration of density was found to be very small with most of the substances, the specific gravity was determined before the stretching had commenced, and again after the wire had been considerably elongated; the density for intermediate amounts of stretching was calculated on the assumption that the change of density is proportional to the increase of length.

In the second method the wires were kept during the whole period of observation in the air-chamber; the hook on the pulley was connected by a fine copper wire, with a brass sliding piece provided with an index and capable of free vertical motion up and down a wooden scale. The lower extremity of the sliding piece was slightly weighted, so as to keep the fine copper wire, which had been previously stretched very nearly to breaking, perfectly straight. The hook on the pulley was also connected by a chain with a lever which served to elongate the wire. On commencing the experiments the weight of the lever was removed from the wire and the resistance of the latter was determined; the position of the index was then noted and afterwards the lever was used to produce the required extension, when again the wire was relieved from stress and after a few minutes the resistance and length of the wire were redetermined. In this way an alteration of length not exceeding $\frac{1}{5000}$th of the whole could be readily measured, but as it was ascertained that in the case of certain wires it was necessary *to remove entirely even the slightest constraint*, such for instance as would be caused by the small weight of the pulley or of the sliding piece, before they would attain in a sufficiently short time their ultimate resistance, a third method was adopted as follows:—

The length of the wire to be examined having been measured as in the first method, it was placed, together with the comparison-wire, at full length in a long wooden box made for the purpose, the two wires having previously been wrapped as usual in paper or surrounded by caoutchouc tubing, and after a sufficient time had elapsed, usually about 15 minutes, to enable them to assume their ultimate ratio of resistance, this latter was determined. The wires were then removed from the box, and the one to be tested stretched as in the first method to a certain extent; they were then replaced, and after the proper time their resistance ratio redetermined. At each removal of the wires from the box the connexions with the other parts of the bridge had to be disturbed; but it was ascertained, as indeed might have been expected from the mode of experimenting,* that this did not in the least affect the value of the ratio of the resistances of the two wires. A few examples will suffice to show the nature of the results obtained.

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* I have frequently removed and replaced the connexions of the wires without causing any alteration of resistance which would amount to $\frac{1}{10000}$th of the whole.

Experiment XXVIII.

A wire of annealed silver, 8 feet in length and ·085 centim. in diameter, stretched by loading.

Load in kilogs.	Total increase of length per cent. produced by the load = A.	Total increase of specific resistance per cent. produced by the load = B.	Ratio of B : A.
5	1·15		
6	3·64	+·034	·0094
6¾	7·28	+·274	0376
7¼	9·38	+·246	0262

Experiment XXIX.

A wire of annealed copper, 8 feet in length and ·095 centim. in diameter, stretched by hand.

Total increase of length per cent. produced by stretching = A.	Total increase of specific resistance per cent. produced by stretching = B.	Ratio of B : A.
2·42	+0·395	·163
4·84	+0·649	·134
11·33	+0·373	·030
23·36	+0·296	·013

Experiment XXX.

A wire of annealed iron, 8 feet in length and ·092 centim. in diameter, stretched by means of a lever, *the stress produced by the weight of the lever being allowed to remain after the permanent extension had been completed.*

Total increase of length per cent. produced by stretching = A.	Total increase of specific resistance per cent. produced by stretching = B.	Ratio of B : A.
3·75	+·129	·034
7·96	·224	·028
11·85	·384	·032
15·74	·640	·041

Experiment XXXI.

A wire of annealed iron, 8 feet in length and ·092 centim. in diameter, stretched by hand.

Total increase of length per cent. produced by stretching = A.	Total increase of specific resistance per cent. produced by stretching = B.	Ratio of B : A.
5·74	−·105	−·0183
7·51	−·143	−·0190
9·49	−·156	−·0164
11·56	−·177	−·0153
16·36	−·150	−·0091

Experiment XXXII.

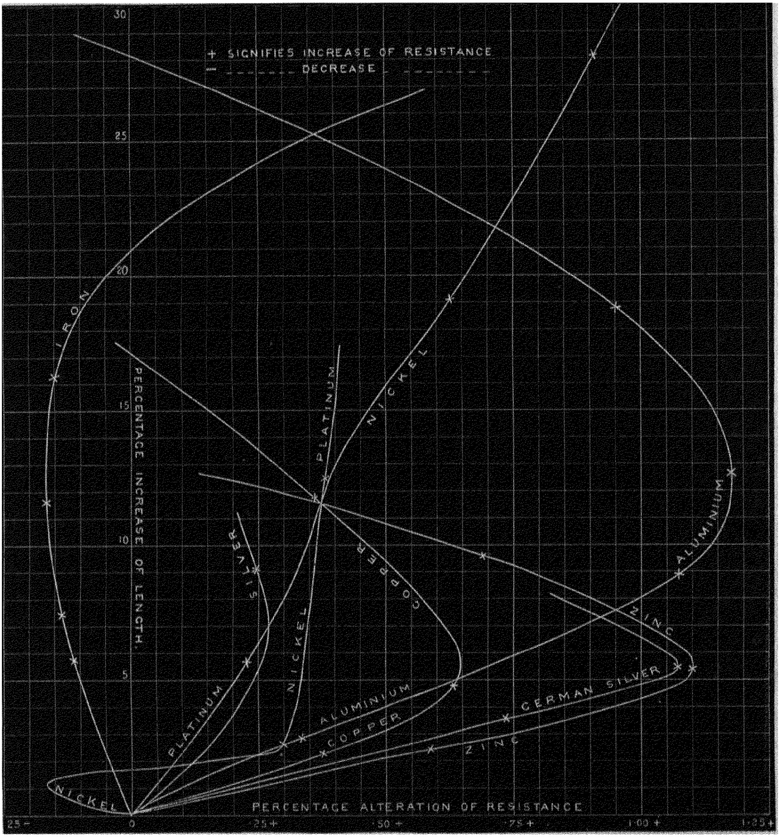
A wire of annealed nickel, 8 feet in length and ·106 centim. in diameter, stretched by loading.

Load in kilogs.	Total increase of length per cent. produced by the load = A.	Total increase of specific resistance per cent. produced by the load = B.	Ratio of B . A.
12	0·32	−0·757	−2 366
18	1·67	+1·260	0·755
20	3·06	+1·514	0·494
22	4·82	+2·052	0·426
24½	7·14	+3·634	0·509

It will be observed that of the four metals—silver, copper, iron and nickel—the two former are at first increased in specific resistance, and that the increase only continues up to a certain amount of stretching when, having reached a maximum, it begins to decrease; whilst with the two latter the first alteration of resistance is in the way of a decrease, which also after attaining a maximum begins to diminish, and with nickel there is soon a comparatively large increase. Of nine metals which have been examined, iron and nickel are the only two which in the first instance show a decrease of specific resistance, whilst the remaining seven are similar to copper and silver in respect to the change of specific resistance due to permanent extension. In the following table the effects of different amounts of permanent longitudinal extension on the specific resistance of copper, zinc, German-silver, nickel, platinum, iron, aluminium and silver are shown by a series of curves. The abscissæ of these curves represent the percentage alteration of specific resistance, and the ordinates the percentage alteration of length, the former on a scale of ·0125 percentage alteration to the millimetre, and the latter on a scale of ·25 percentage alteration to the millimetre;

with nickel, however, the alteration of resistance is so very much larger in proportion to increase of length than is the case with the other metals, that the abscissæ are represented on a scale of ·05 percentage of alteration to the millimetre, and the ordinates on a scale of ·0625 percentage alteration to the millimetre.

TABLE XV.—Curves showing the permanent alteration of specific resistance produced by permanent longitudinal extension.



Remarks on Table XV. and Experiments XXVIII. to XXXII. inclusive.

It should be stated that before making any observation of the resistance or length, the wire was stretched about 2 per cent. of the original length in order to render it sufficiently straight, and that the resistance and length after this stretching are taken as the starting points from which the percentage alterations are measured; in the cases, however, of German-silver and nickel this was not done, as they were deemed to be sufficiently straight without stretching, and, moreover, not capable of much extension before breaking.

The curves are evidently of parabolic shape, except perhaps in the case of nickel, whose behaviour seems to be quite abnormal. With this metal the curve after passing to the left, showing that the specific resistance is diminished by extension, takes a sharp turn round to the right, and, after passing almost vertically upwards, makes another bend afterwards, proceeding almost in a straight line. Experiment XXXII., as well as the curve for nickel also, show how much larger is the alteration of specific resistance, whether decreasing or increasing, in comparison to the alteration of length, than is the case with any of the other metals.

By comparing Experiment XXX. with Experiment XXXI., we see the effect of leaving a load on the wire after the extension has taken place; the decrease of specific resistance of iron produced by a moderate amount of extension is in this case replaced by an increase. Now we have seen in Part I. that the wire will not quite assume the length which it would have if entirely relieved from stress, but the change in resistance is much larger than can be accounted for by any change of dimensions, and in some instances the effect of a much slighter restraint than was produced by the stress of the lever sufficed to make a very appreciable difference in the specific resistance. The silver wire formed a remarkable example of the kind; with this wire the specific gravity was determined after each stretching, and the resistance was measured both before and after the former operation, the only load left on the wire being that of the pulley, which produced a stress of a quarter of a kilogramme on each half of the wire. The specific resistance after the different amount of stretchings recorded in Experiment XXVIII. was decreased ·03, ·27, ·28 and ·2 per cent. by merely removing the pulley and taking the wire down for the purpose of weighing it in water. These alterations are, it is true, not absolutely large, but if we compare them with the total alteration of specific resistance given in the third column* of the experiment, we see that they are relatively very considerable. Moreover, in this case not the slightest alteration in the length of the wire caused by the removal of the pulley could be detected; neither was the change brought about by the weighing in water, since a similar alteration was caused when the pulley was simply removed and replaced without any such weighing; nor, again, was it due to the restitution of conductivity, which we shall learn presently that rest causes after strain, since the resistance was decreased by the removal and replacing of the pulley in one experiment 14 hours after the permanent extension had been made; a much longer time, of course, elapsing in this case than was required in the above mentioned operations.

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* The values given in this column are calculated from the resistance determined *after the pulley had been removed and then replaced.*

Again, though the curves are not capable of showing any sudden changes in the ratio of the alteration of the specific resistance, and the extension at the two critical points before alluded to, inasmuch as with most metals these points occur when a comparatively small amount of extension has taken place, yet it will be seen from Experiment XXVIII. that with the increase of length caused by the load of 6¾ kilogs. there is a sudden increase of the ratio B : A, and this load corresponds very closely with that at the second critical point of silver.

Hammering.

W. THOMSON, in 1857, experimented on the effects of hammering and permanent extension on the electrical conductivity of copper, and though no actual numbers are given in his paper, states* that "the greatest degree of brittleness produced by tension does not alter the conductivity of the metal by as much as one-half per cent." He, moreover, adds: "A similar experiment showed no more sensible effect on the conductivity of copper wire to be produced by hammering." The foregoing experiments, it will be seen, fairly bear out THOMSON'S statement with reference to the small amount of alteration produced in the specific resistance of copper by longitudinal extension, and show a still smaller change in the resistances of silver and platinum. But since these experiments at the same time showed that the alteration after increasing to a maximum in one direction began to decrease, and in certain cases finally set in in the opposite direction, it seemed desirable to ascertain whether hammering would produce like effects on the specific resistance.

The third of the methods used in determining the change of specific resistance by extension was here employed, except that the wires were now lengthened by hammering them transversely. The last process was accomplished rather by a great number of comparatively small blows than by a less number of large ones, so as to hammer the wire as uniformly as possible throughout its whole length. The following are examples of the results arrived at:—

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* Proc. Roy Soc., vol. viii., p. 553.

Experiment XXXIII.

A wire of annealed copper, 8 feet in length and ·095 centim. in diameter, hammered transversely throughout its entire length.

Total increase of length per cent produced by hammering = A.	Total increase of specific resistance per cent produced by hammering = B.	Ratio of B : A.
1·89	+·017	·009
3·11	+·174	·056
5·95	+·009	002
9·66	—·209	—·022
16·76	— 530	— 032

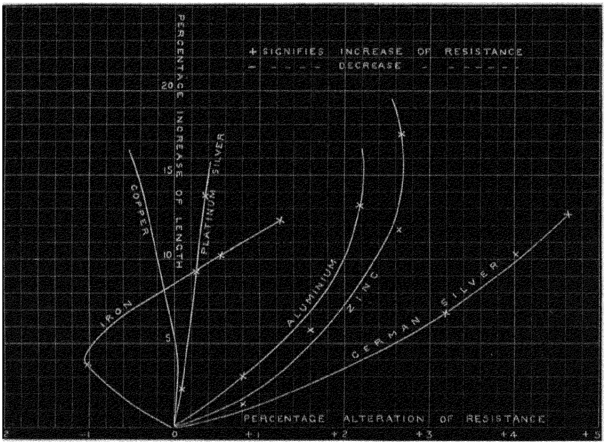
Experiment XXXIV.

A wire of annealed iron, 8 feet in length and ·092 centim. in diameter, hammered transversely throughout its entire length.

Total increase of length per cent. produced by hammering = A.	Total increase of specific resistance per cent. produced by hammering = B.	Ratio of B A
2 04	—0 082	—·040
3 94	—1 035	— 263
10·27	+0 554	+ 054
12·31	+1·283	+ 105

In the next table will be found a series of curves showing the change of specific resistance produced by hammering on the metals iron, copper, zinc, aluminium, German-silver, and platinum-silver. The abscissæ showing the percentage alteration of specific resistance are on a scale of ·05 per cent. for 1 millim., and the ordinates representing the increase of length per cent. are on a scale of ·25 per cent. for 1 millim.

TABLE XVI.—Curves showing the alteration of specific resistance produced by hammering.



Observations on the curves in Table XVI.

Experiments XXXIII., XXXIV., and Table XVI. show that the effect of hammering on the specific resistance is of a somewhat similar character to that of permanent extension. With all the metals, except iron, the specific resistance is at first increased, and this increase, after reaching a maximum, begins to diminish, but with iron the first effect is decrease of resistance which also, after attaining a maximum as the hammering is carried to a greater and greater extent, begins to diminish until finally there is a comparatively large increase of specific resistance.

The changes produced, however, by hammering, though similar in kind to those produced by longitudinal extension, are very different in amount, and a comparison of the two sets of curves and the scales on which they are formed shows that the alterations in the former case are very much greater than those in the latter.

The neutral points also, *i.e.*, the points where the curves cut the axis of ordinates, are different in the two tables, being for copper and iron much higher for the extension than for the hammering, but with zinc and German-silver lower for the extension than the hammering.

Torsion.

As it seemed desirable to supplement the observations of extension and hammering with others on torsion, with the view of ascertaining whether the strain caused by twisting a wire beyond the limits of elasticity would at all resemble the effect produced on the specific resistance by strain set up by the two former processes, a few experiments were made in which the wires having been secured at one end and stretched sufficiently to make them tight, were twisted more and more until they broke.

Experiment XXXV.

A wire of annealed zinc, 8 feet in length and ·095 centim in diameter, twisted.

Amount of torsion; the torsion of a complete revolution in a length of 1 centim. taken as unit = A.	Total increase of specific resistance per cent. produced by the torsion = B.	Ratio of B : A.
·083	+ ·269	3·24
·167	+ ·546	3 27
·250	+ ·376	1·27

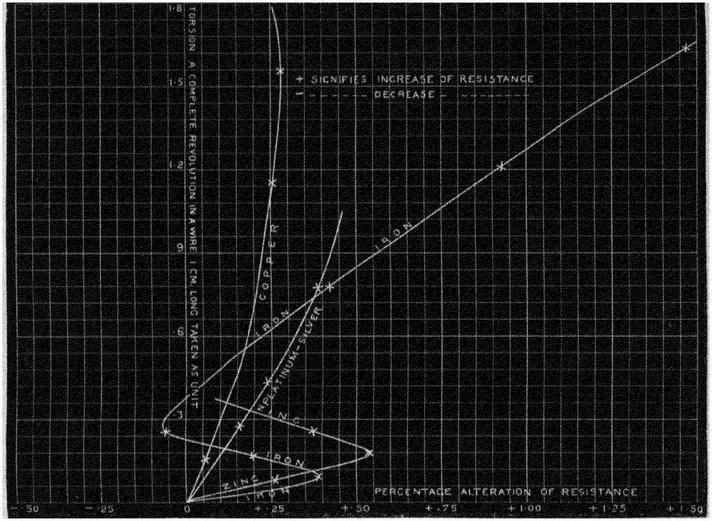
Experiment XXXVI.

A wire of annealed iron, 8 feet in length and ·092 centim. in diameter, twisted.

Amount of torsion; the torsion of a complete revolution in a length of 1 centim. taken as unit = A.	Total increase of specific resistance per cent produced by the torsion = B. – signifies decrease of specific resistance	Ratio of B : A.
·080	+ 394	+4 93
160	+ ·195	+1 22
·240	– ·053	–0 22
·320	– 076	–0·24
·800	+ 413	–0 52
1·200	+ ·928	–0 73
1·600	+1 454	–0 91

In Table XVII. are drawn a series of curves showing the changes produced by permanent torsion in the specific resistance of zinc, iron, copper, and platinum-silver. The abscissæ in these curves represent the percentage alteration of resistance on a scale of ·0125 to 1 millim., and the ordinates representing the torsion in terms of the torsion of a complete revolution in a length of 1 centim. of the wire taken as unit on a scale of ·015 to 1 millim.

TABLE XVII.—Curves showing the permanent alteration of specific resistance produced by permanent torsion.



Remarks on Table XVII.

The effect of torsion on the specific resistance is evidently of a similar nature to that of the effect of permanent extension and hammering, but the amount of alteration shown at the turning point of the curves is in all cases much less than that observed at the turning points of the curves exhibiting the results of hammering and extension. In the case of iron, the first torsion applied increases the specific resistance, but further torsion acts in the same direction as the strain caused by hammering and extension, *i.e.*, diminishes the specific resistance up to a certain point, but beyond this point increases it. It may be that the increase produced in iron by the first few turns is due to magnetic influence on the resistance, for as THOMSON has observed,* the electric current instead of flowing rectilineally along the wire would flow in helical lines, and would therefore increase the resistance by longitudinal magnetization.†

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* W. THOMSON, "Electrodynamic Qualities of Metals," Phil. Trans. 1879, Part I., § 229.

† W. THOMSON, "Electrodynamic Qualities of Metals," Phil. Trans. 1856. Bakerian Lecture, § 146, and H. TOMLINSON, Proc. Roy. Soc., June 17, 1875. [Note added May, 1882.—I have since found that *very slight* extension increases the specific resistance of iron, and am therefore inclined to reject the above hypothesis, and to believe that *very slight* strain of any kind increases the specific resistance.]

It will be noticed also that the curves like those of Tables XV. and XVI. are of parabolic shape.

THE ALTERATION OF SPECIFIC GRAVITY PRODUCED BY PERMANENT EXTENSION, HAMMERING, AND PERMANENT TORSION.

In calculating the values of the specific resistance after the wire had been subjected to any one of the above-mentioned processes, it was in the first instance assumed that there was no change in the specific gravity of the substance, and afterwards a correction applied for such change. The specific gravity of the wires in the annealed condition having been determined, the change wrought by extension, hammering, or torsion carried to an extent which was about three-fourths of that which the substance would bear without rupture was determined by a very delicate balance, and though it was not possible to dislodge any air bubbles which might be attached to the metal when immersed in water by boiling for fear of partially annealing the specimens, these bubbles were brushed off very carefully and considerable pains were taken in the weighings. A very fine platinum wire was used for the purpose of suspending the substances in water, and the vessel employed for holding the water was of considerable size so as to avoid the necessity of bending the wire to any great extent. The coils in which the wire under examination was wound, when weighed, were kept together by fine copper wire, and the weight of this in air and water together with that of the fine platinum wire was from time to time determined. The next experiments show that the change in specific gravity is roughly proportional to the amount of strain, provided this strain is not continued up to the point of rupture.

Experiment XXXVII.

Wire of annealed silver, 8 feet in length and ·090 centim. in diameter, stretched by loading.

Increase of length per cent. = A.	Decrease of specific gravity per cent = B.	Ratio of B A.
1 15	0124	011
3·64	0461	013
7·28	0892	012
9 38 (broke)	·1465	·016

Experiment XXXVIII.

A wire of annealed copper, 3 feet in length and ·095 centim. in diameter, twisted.

Number of complete revolutions of torsion = A.	Decrease of specific gravity per cent = B.	Ratio of B : A.
400	1·16	·0029
600	1·66	·0028
800	2·13	·0027

In Table XVIII. will be found the extent of change in the specific gravity of the different metals produced by a given amount of stretching, hammering, and twisting.

TABLE XVIII.

Name of metal.	Percentage alteration of specific gravity attending a permanent increase of length of 1 per cent. produced by stretching.	Percentage alteration of specific gravity attending a permanent increase of length of 1 per cent. produced by hammering. + signifies increase of specific gravity.	Percentage alteration of specific gravity attending a permanent torsion equal to that of 1 complete revolution in a length of 1 centim + signifies increase of specific gravity.
Platinum	— ·0620		
German-silver	— ·0612	— ·0136	
Nickel	— ·0510		
Zinc	— ·0509	— ·0146	— ·01212
Iron	— ·0203	— ·0215	— ·00283
Silver	— ·0156		
Aluminium	— ·0082	— ·0183	
Copper	— ·0178	+ ·0065	— ·00252
Platinum-silver		+ ·0159	+ ·00190

In the last table a — sign signifies a decrease and a + sign an increase of specific gravity.

The specific gravity of all the annealed metals here examined is decreased by permanent extension.*

Hammering also decreased the specific gravity of all the metals subjected to this process except copper and platinum-silver; with these metals the specific gravity was slightly increased.

Permanent torsion decreased the specific gravity of copper, zinc, and iron, and increased that of platinum-silver.

Of the three processes, torsion produced the greatest maximum change in the specific gravity, thus it will be seen from Experiment XXXVIII. that the specific gravity of copper was decreased by the torsion more than 2 per cent.—a large amount considering the small alteration* which extension and hammering can produce.

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* This apparently is not always the case, as Sir W. THOMSON mentions in his article on "Elasticity," 'Brit. Encyc.,' p. 1, that a certain specimen of copper wire annealed in hot sand had its density *increased* more than 1 per cent. by longitudinal extension.

The results recorded in the last table enabled the previously mentioned correction for change of specific gravity to be made in calculating the specific resistance.

For if Δ_1 and Δ_2 be the specific gravities respectively before and after stretching, hammering, or twisting, and S and x be respectively the specific resistances uncorrected and corrected for change of specific gravity, $x = \mathrm{S} \times \frac{\Delta_1}{\Delta_2}$.

Now the table furnishes the means of determining

$$\frac{\Delta_1 - \Delta_2}{\Delta_1} = \alpha \text{ say ;}$$

and

$$\frac{\Delta_1}{\Delta_2} = \frac{1}{1-\alpha} = 1 + \alpha \text{ very nearly,}$$

since α is very small.

Therefore

$$x = \mathrm{S} \times (1 + \alpha).$$

The correction though small was applied in all cases.

EFFECT OF COOLING SUDDENLY ON THE SPECIFIC RESISTANCE OF STEEL.

We have seen in Part I. that the effect of suddenly chilling steel heated to a high temperature produces a somewhat similar effect on the elasticity to that of excessive permanent extension, and it was concluded to be highly probable that whether the distance between the molecules be increased by mechanical strain or by the strain caused by sudden cooling, the elasticity in the direction of the line of separation of the molecules diminishes to a minimum as the separation increases, and then begins to increase. Now BARUS† has proved that the specific resistance of steel *increases* continuously with its hardness, but BARUS's experiments were made with steel heated at or above a visible red, and as the strain produced by extension, hammering, and torsion had been shown to produce up to a certain point *decrease* of resistance, it seemed a matter of some interest to ascertain whether heating the steel to a lower temperature than that of dull red and then cooling slowly would not also have the effect of decreasing the specific resistance. The following experiment was therefore tried :—

---

* Not so much as 1 per cent. in any case which I have examined.

† Phil. Mag., November, 1879.

Experiment XXXIX.

A piece of annealed piano-steel wire, 8 feet in length and ·083 centim. in diameter, cooled suddenly by plunging it into cold water after it had been heated to various temperatures.

Condition.	Number proportional to specific resistance.*
Soft.	91168
Heated and cooled, hiss not audible	91095
Heated and cooled, hiss just audible.	91094
Heated below dull red but hiss very audible on cooling	91118
Heated to dull red and cooled.	91891

In this last experiment the wire was tempered when coiled, the coils being held together by wrapping fine iron wire round them, and passing a burner rapidly round the coils until it was supposed that the requisite temperature had been applied, when the wire was suddenly plunged into water at a temperature of 10° C.

This experiment, though rough as regards the mode of tempering, shows plainly that the specific resistance is *decreased* by the sudden cooling until the tempering is performed at some temperature under dull red, when the specific resistance begins to increase. It will be shown also in Part IV. that the thermo-electric properties of steel are affected in precisely the same manner, that is, that tempering beyond a dull red temperature produces opposite effects to tempering under a dull red temperature.

THE RECOVERY OF ELECTRICAL CONDUCTIVITY PRODUCED BY TIME IN WIRES WHICH ARE IN A STATE OF STRAIN.

In all the experiments described in this Part it was observed that when the wires had been subjected to stresses of any kind, whether purely mechanical or otherwise, which sufficed to produce permanent strain, they invariably gained in electrical conductivity when allowed to rest. The amount of decrease of resistance produced by rest varied however considerably with different metals, being very conspicuous in German-silver and hardly perceptible with platinum-silver.

Table XIX. shows the influence of rest in restoring the electrical conductivity of wires of German-silver, copper, iron, zinc and platinum-silver after hammering so as to increase the length about 15 per cent.; the observations being commenced 20 minutes after the hammering had been completed.

The abscissæ of the curves represent the decrease of resistance on a scale of ·0085 per cent. to 1 millim., and the ordinates on a scale of one hour to the millimetre.

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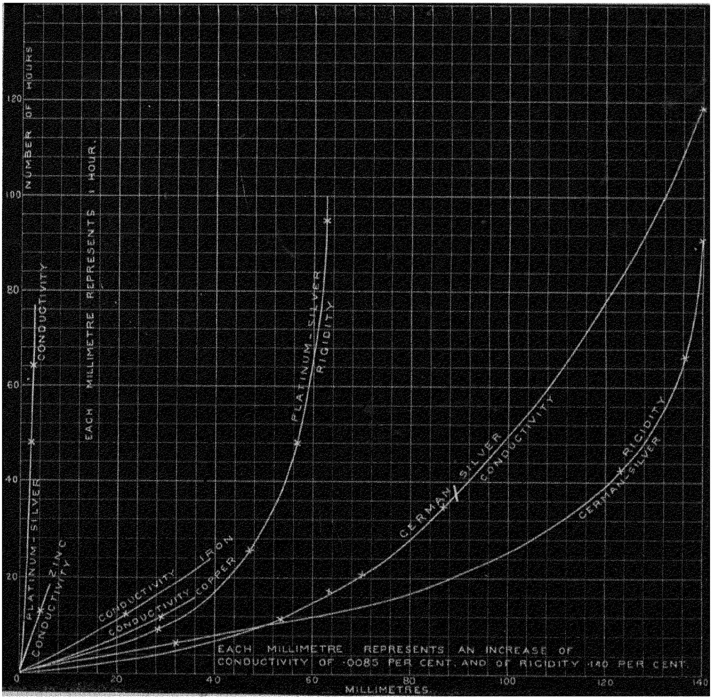
* The resistance of the comparison-wire is here taken as unit.

The tables also show a curve representing the restitution of torsional rigidity in German-silver after hammering, the abscissæ representing the increase of rigidity on a scale of ·0085 per cent. to 1 millim. and the ordinates on the same scale as for the other curves.

TABLE XIX.—Curves showing the recovery of electrical conductivity and of torsional rigidity produced by time in the case of wires which have been hammered transversely.



This table shows the marked difference between German-silver and platinum-silver in respect to the restitution of conductivity produced by rest on these metals when in a state of strain, and this large difference is not confined to strain produced by hammering, but was found to exist also in strain produced by extension, torsion, and wire-drawing, and when taken in combination with the results given in Tables XV., XVI., and XVII. on the alteration of specific resistance produced by strain, shows most conclusively the superiority of platinum-silver to German-silver in making standard resistance coils.

We have seen also in Part I. that rest, materially in some cases and to a certain extent in all, increases the elasticity of metals; now German-silver wire furnishes a conspicuous example of this; and the curve in the table representing the increase of torsional rigidity produced by rests, shows plainly that the restitution of elasticity and electrical conductivity go hand in hand. And this circumstance, when taken in conjunction with the fact that there is no change in the dimensions of the wire which would at all account for the increase of elasticity or conductivity, evidently teaches us that when we can increase the elasticity without altering the mean molecular distance, we at the same time increase the electrical conductivity.

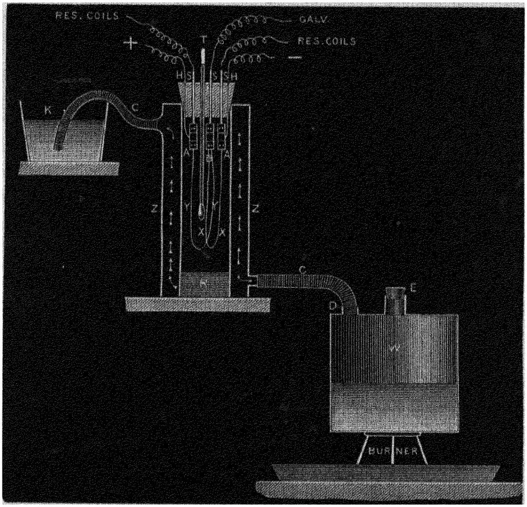
THE INFLUENCE OF PERMANENT STRAIN ON THE CHANGE OF ELECTRICAL CONDUCTIVITY PRODUCED BY ALTERATION OF TEMPERATURE.

Permanent extension, hammering, and torsion.

The previous observations on the change of specific resistance produced by extension, hammering, and torsion having given rise to the suspicion that these processes caused the wires to alter their susceptibility to change of resistance from change of temperature, and that this alteration of susceptibility bore a marked relation to the alteration of the thermo-electrical properties which are also caused by stress and strain, it was considered very advisable to further investigate the matter.

In the first few trials the large air chamber previously described was employed, but the changes wrought in the coefficient of increase of resistance from rise of temperature by the various strains were found to be so small that special precautions had to be taken to secure reliable results. Accordingly the following apparatus was made use of. In the figure Z Z is an air chamber made of two concentric brass tubes 12 inches in length. The diameter of the wider tube is 4½ inches, and of the inner tube 2½ inches, so that the two enclose between them a layer of air 1 inch thick. The two tubes are connected at the top and bottom by brass rings soldered to them, and two pieces of brass tubing soldered into the outer cylinder, one near the bottom and the other near the top, communicate by means of indiarubber tubing with a boiler, W, and a vessel of cold water, K, respectively, so that steam from the boiler can fill the entire space between the two concentric cylinders and be condensed in K. Three stout wires of copper, S S S, 4 inches in length, pass through a cork which fits tightly the inner cylinder, and their upper ends are connected with the resistance coils and galvanometer, whilst to their lower ends are attached three clamps, A A and B. To the clamps are soldered silk-covered copper wires, which, passing through the cork, are connected with the poles of the battery. The ends of the wire to be tested, X, and the comparison-wire, Y, are secured to the clamps at A A and B, the two wires having previously to clamping been enveloped in cotton tubes of the same lengths as the wires. As a further precaution, the clamps A were well wrapped up in silk, and with the wires were enclosed in a cardboard cylinder which fitted neatly into the inner brass cylinder. When it was necessary to heat the wires, jackets of several folds of baize were placed round the outside of the air chamber and also completely covered the top.

Fig. 17.



The connexions with the resistance coils of 100 ohms and the galvanometer were exactly as in previous experiments, so that from the disposition of the wires as shown in the figure, it will be seen that the influence of change of temperature on the connexions would be entirely neglectable. Moreover, the boiler and air chamber, which were some distance from the box of resistance coils, were screened from the latter, which was covered on the outside with tin foil, and only opened at the moments of actual testing for the balance between X and Y. The wires X and Y were in all cases 50 centims. long, and so disposed that they occupied the central portion of the chamber; whilst a thermometer, T, served to indicate the temperature. The air chamber stood upright on a table, and the lower portion of it, R, was packed with sawdust on which the base of the cardboard cylinder rested.*

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* It would have been better to use a cylinder of copper foil instead of the cardboard, as thereby the temperature would have been rendered more uniform throughout the length of the wire.

The mode of experimenting adopted was as follows: X and Y having been placed in position and the air chamber covered with its baize jackets, the balance between the two wires was observed, and the thermometer having been pulled sufficiently far out of the cork to enable one to take the reading of the temperature, was afterwards replaced. The water in W was then boiled, and in about half an hour the air in the chamber was found to be at 100° C., and was allowed to remain so for another half an hour, when X and Y were again balanced. The burner was next taken from under W, when a vacuum was formed in the upper part of the boiler by the steam condensing; this vacuum was at once filled by the atmospheric pressure forcing water from K, and when the space between the two brass cylinders had in a short time become filled with the water from K, the cork E was removed from the boiler, and a siphon action allowed to continue from K which was kept supplied with cold water. The jackets were then removed, and the cork E having been replaced, the whole arrangement was suffered to rest for two hours, when the thermometer indicating the temperature of the air chamber to be within a degree or so of the original temperature, the balance between X and Y was once more determined. As the processes of hammering and stretching by increasing the length of X might possibly have caused an error by making the lower portion of the wire to occupy a lower position in the chamber than the corresponding portion of Y,[*] the former wire was from time to time shortened to the same length as the latter. Also, since with all the wires it was found impossible, even by using the greatest care in annealing, to find two pieces of the same wire which would agree exactly in their co-efficients of increase of resistance, these were compared before X was subjected to strain of any sort. The following experiments illustrate the nature of the results obtained:—

Experiment XL.

An annealed copper wire, ·095 centim. in diameter, was stretched by hand permanently to different extents, and after each stretching tested for alteration of resistance from change of temperature.

Percentage increase of length.	Percentage of average temporary superior increase of resistance of stretched wire over unstretched for 1° C. between 20° C and 100° C + signifies superior increase of stretched wire on rise of temperature	Percentage of average permanent decrease of resistance for 1° C. between 20° C. and 100° C of stretched wire.
5 83	— 00046	00104
11·40	—·00074	00107
22·27	—·00045	·00354
32 73	+·00378	00190

---

[*] I have found it very difficult even with such an arrangement as the above to get an *exactly* uniform temperature, except in the central portion of the chamber.

Experiment XLI.

An annealed iron wire, ·0065 square centim. in section, stretched by hand.

Percentage increase of length.	Percentage of average temporary superior increase of resistance of stretched wire over unstretched for 1° C between 20° C. and 100° C. + signifies superior increase of stretched wire on rise of temperature.	Percentage of average permanent decrease of resistance for 1° C between 20° C and 100° C. of stretched wire, − signifies permanent increase.
6·0	+·0071	−·00165
12·0	+·0055	+·00230

Experiment XLII.

An annealed copper wire, ·095 centim. in diameter, hammered.

Percentage increase of length.	Percentage of average temporary superior increase of resistance of hammered wire over unhammered for 1° C between 20° C. and 100° C. + signifies superior increase of hammered wire on rise of temperature.	Percentage of average permanent decrease of resistance of hammered wire for 1° C. between 20° C. and 100° C.
1·74	−·00340	00836
5·81	+ 00117	00603
11·05	+·00122	·00447
21·36	+·00139	·00072

Experiment XLIII.

An annealed iron wire, ·063 centim. in diameter, hammered.

Percentage increase of length.	Percentage of average temporary superior increase of resistance of hammered wire over unhammered for 1° C. between 20° and 100° C. + signifies superior increase of hammered wire on rise of temperature.	Percentage of average permanent decrease of resistance of hammered wire for 1° C. between 20° and 100° C.
6·13	+·0009	·0033
10·72	+·0000	·0056

Experiment XLIV.

An annealed copper wire, ·095 centim. in diameter, permanently twisted.

Torsion in terms of a unit taken as the torsion in a wire 1 centim. long when twisted through one revolution.	Percentage of average temporary superior increase of resistance of twisted wire over untwisted for 1° C. between 20° and 100° C. + signifies superior increase of twisted wire on rise of temperature.	Percentage of average permanent decrease of resistance of twisted wire for 1° C. between 20° and 100° C.
0 652	−·000797	
1·087	−·000824	·00003
3·261	−·000977	·00015
6·520	+·002097	−·003074

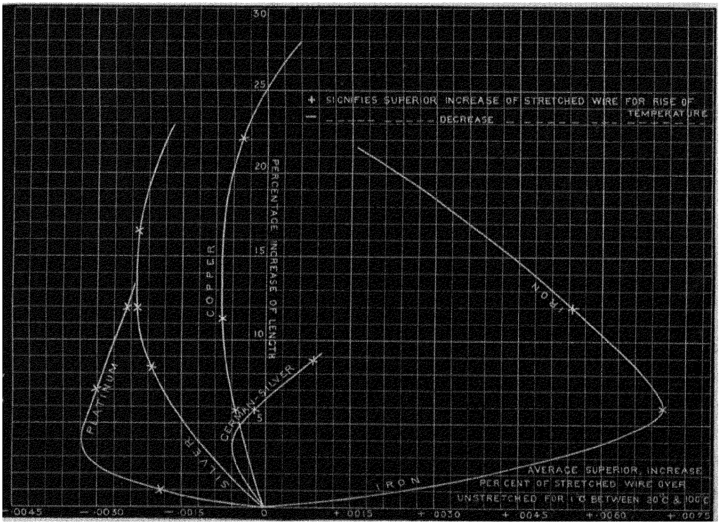
Experiment XLV.

An annealed iron wire, ·063 centim. in diameter, permanently twisted.

Torsion in terms of a unit taken as the torsion in a wire 1 centim. long when twisted through one revolution.	Percentage of average temporary superior increase of resistance of twisted wire over untwisted for 1° C. between 20° and 100° C. + signifies superior increase of twisted wire on rise of temperature.	Percentage of average permanent decrease of resistance of twisted wire for 1° C. between 20° and 100° C.
0·260	+·00474	·00025
1·090	+·00421	·00050
2·180	+·00092	·00100

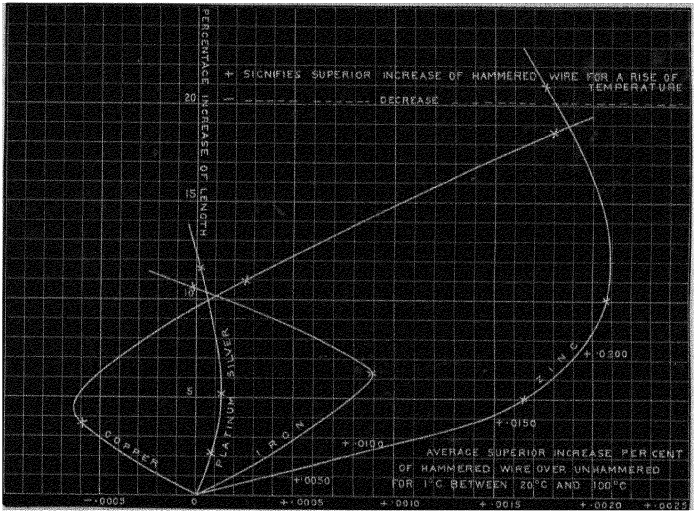
The effect of permanent extension, of hammering, and of torsion on the alteration of resistance produced by change of temperature in the case of copper, of iron, and of other metals is shown in Tables XX., XXI., and XXII.

TABLE XX.—Curves showing the effect of permanent extension on the temporary alteration of electrical resistance produced by change of temperature.



In Table XX. the curves have their abscissæ representing the average superior increase of resistance for a rise of 1° C. of the stretched wire on a scale of ·000075 percentage of superior increase for each millimetre, and in Table XXI. the curves are drawn with their abscissæ representing the average superior increase of resistance of the hammered wire for a rise of 1° C. on a scale of ·000025 percentage of superior increase for each millimetre. In both sets of curves the ordinates represent the percentage of permanent increase of length on a scale of ·25 percentage of increase of length for 1 millim.

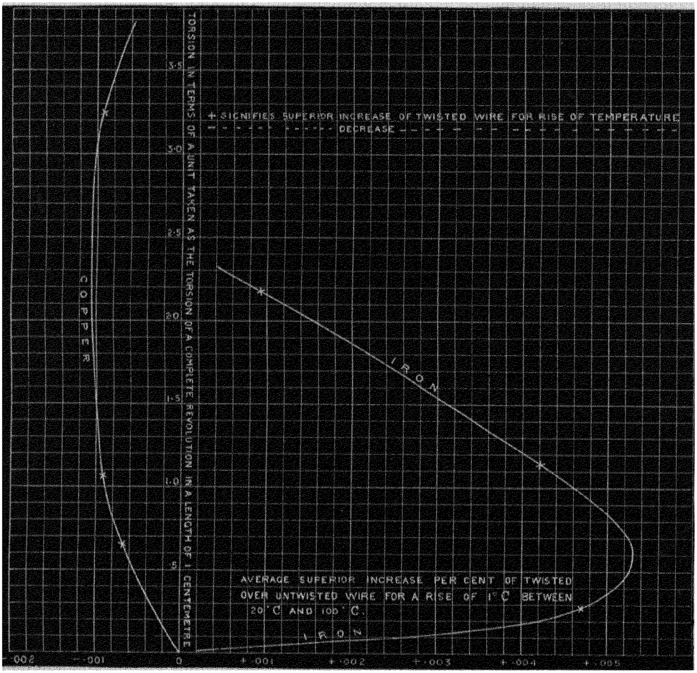
Table XXI.—Curves showing the effect of hammering on the temporary alteration of electrical resistance produced by change of temperature.



For zinc the abscissæ represent 10 times as much alteration of resistance as they do for the other metals.

In Table XXII. the abscissæ of the curves represent the superior increase of resistance of the twisted wire on a scale of ·0005 percentage of superior increase for each millimetre, and the ordinates the amount of permanent torsion in terms of a unit taken as the torsion of a complete revolution in a length of 1 centim. on a scale of ·025 unit for 1 millim. In all the tables a + sign before an abscissa signifies that the strained wire increases most on rise of temperature, and the values of these abscissæ were determined as follows:—Let $\frac{A}{B}$ and $\frac{A'}{B'}$ represent the ratios of the wire to be tested, and the comparison-wire at any two temperatures t and t'; then the average percentage superior change of resistance is $\frac{\left(\pm\frac{A'}{B'}\mp\frac{A}{B}\right)\times 100}{(t'-t)\times\frac{A}{B}}$, t' being the higher of the two temperatures, which were about 100° C. and 20° C.

TABLE XXII.—Curves showing the effect of permanent torsion on the temporary alteration of electrical resistance produced by change of temperature.



Remarks on the last experiments and tables.

The first point to be noticed is that the metals examined may be divided into two classes. In the first of these classes, which includes iron, zinc and platinum-silver,* the strained wire is *most increased in resistance* by rise of temperature up to a certain limit of straining; whilst beyond this limit further strain diminishes the first effect. In the second class, which comprises copper, silver, platinum and German-silver, the strained wire is *least increased in resistance* by rise of temperature, but that, here again, after a certain limit of strain has been reached the first effect begins to be diminished. Now it will be shown in Part IV. that the metals of the first class, when subjected to a moderate amount of strain, whether the latter be produced by permanent extension, hammering, or torsion, or by other than purely mechanical means, are *thermo-electrically positive* to pieces of the same metal unstrained, and that the first effect is ultimately reversed if the strain be carried to excess; whilst the metals of the second class are, when moderately strained, *thermo-electrically negative* to unstrained pieces of the same metal, but when subjected to excessive strain become thermo-electrically positive. We are therefore led to the conclusion that there must be some close relationship between the thermo-electrical properties of metals, and their temporary alteration of resistance from change of temperature, and that *strain renders a piece of metal thermo-electrically positive or negative to a piece of the same metal unstrained according as the strained piece is caused to be less or more increased in electrical resistance by rise of temperature.*

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* In some experiments made by H. A. TAYLOR (see "Report of Electrical Standards Commission," Appendix II., Brit. Ass., York Meeting, 1881), the effect of drawing platinum-silver wire to finer and

Another point to be considered is that the metal which is *increased* in specific resistance by the strain is not always rendered *less* susceptible to change of resistance from alteration of temperature, for, as we have seen, zinc and platinum-silver are increased in specific resistance by moderate strain, and are yet at the same time rendered more susceptible to alteration of resistance from variations of temperature. With the former of the above-mentioned metals, the alteration of susceptibility to change of resistance from change of temperature is comparatively much larger than is the case of any of the other metals; whilst with platinum-silver the effect of strain in this respect is comparatively slight, and of the opposite kind to that which is produced on its two constituents.

Again, the third column in the experiments shows that the permanent effect produced by the change of temperature is not of the same nature with iron as with copper; with the former metal the resistance is in some cases actually increased by the annealing when the strain is moderate, and in those cases where a permanent decrease of resistance is caused this increases with increased strain; with the latter, on the contrary, permanent decrease of resistance after moderate straining is the result of the annealing, and this decrease after increasing to a maximum begins to become less with further strain. The difference between iron and copper in the above-mentioned respect is readily intelligible when we remember that these metals differ as regards the kind of alteration of specific resistance caused by strain, as we have seen that with iron the

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finer gauges was found to be to diminish the temperature coefficient, but it seems that Mr. TAYLOR annealed the wires after the process of drawing.

specific resistance is decreased by moderate strain but increased by excessive strain; whilst with copper, moderate strain effects increase, and excessive strain, decrease of resistance.

THE EFFECT OF TEMPORARY STRESS ON THE ALTERATION OF ELECTRICAL RESISTANCE PRODUCED BY CHANGE OF TEMPERATURE.

As permanent strain had been proved to alter the susceptibility to change of resistance from change of temperature in a manner which suggested an intimate relationship between this susceptibility and the thermo-electric properties of metals, some attempts were made to determine the effect of such temporary stress as could be caused by longitudinal traction, on the alteration of resistance produced by change of temperature, partly with a view of establishing a still closer relationship between the above-mentioned physical qualities, and partly with the object of ascertaining whether the increase of elasticity which WERTHEIM's experiments* seemed to have proved to be produced in iron and steel when the temperature is raised from 20° C. to 100° C., would be rendered manifest in experiments on the influence of stress on the electrical resistance. The difficulties here encountered seemed at first sight to be so great as to render it an almost impossible task to *measure* with any approach to accuracy the effect sought, unless this effect should be something very appreciable compared with the percentage alteration of resistance produced by stress at ordinary temperatures; for, as we have already seen, the increase of resistance produced by raising the temperature to 100° C. would be some hundreds of times greater than any change of resistance which can be produced in most metals by mechanical stress. After several failures, however, these difficulties were overcome, and I succeeded in measuring with almost the same accuracy the comparatively minute alterations of resistance produced by temporary longitudinal traction at the temperature of 100° C. as at the ordinary temperature of the room.

To accomplish the desired object the large air chamber used in the first part of the enquiry was provided with two tubes, one near the bottom and the other near the top; these tubes served the purpose of conveying steam into the bottom of the outer of the two cylindrical chambers of which the vessel consisted, and out again, near the top of the vessel, into a tub of cold water, a hole at the top of the chamber which had been used previously for the purpose of filling the chamber with water having been corked up after the water had been emptied out. The air chamber was wrapped round with several layers of baize, and these in turn surrounded with several layers of paper tied round with string. The top of the air chamber was also well covered in the same manner with baize and paper, and the table on which the air chamber rested was surrounded on all sides with like material with the exception of a small aperture through which the experimenter was enabled to adjust the weights on the scale-pan attached to the wire to be tested. The scale-pan was suspended from the pulley, which was 6 inches from the bottom of the chamber, by a wire sufficiently strong to bear the weights employed, and this wire passed through a hole in the table only just large enough to allow of free motion. The doors and windows of the room were kept shut during the testing at the higher temperature, and the usual precautions were taken to avoid any risk of permanent set or any liability to change of elasticity from testing too soon after permanent extension. In about an hour after the first entrance of the steam into the air chamber the temperature of the air at the top and bottom was nearly, if not quite, at 100° C, but it was found necessary to allow the steam to enter for three or four hours before the wire to be experimented on and the comparison-wire had assumed a sufficiently stable resistance-ratio, and even after this time there would be slow and very minute variations of this ratio first in one direction and then in the opposite. Any errors, however, which would result from slow and minute variations were got rid of in the following manner :—Let a_1, b_1; a_2, b_2, &c., be the apparent alterations produced by putting on and taking off the load several times in succession; then the true alterations due to the load will be very nearly

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* Ann. de Chimie et de Phys., 3^me^ série, 1844, p. 431.

$$\frac{a_1+a_2+2b_1}{4},\ \frac{b_1+b_2+2a_2}{4},\ \&c.$$

The following experiment will show how accurately the measurements could be made even at the temperature of 100° C.

Experiment XLV.

An annealed iron wire, 7 feet in length and ·067 centim. in diameter, was loaded and unloaded several times with a weight of 3 kilogs.; this weight was then removed and a rest of 48 hours allowed, when, on again testing with 3 kilogs., the recovery was found to be quite perfect. The wire was then heated to 100° C., and having been maintained at this temperature for several hours was again tested. Afterwards the air chamber was suffered to cool down to the original temperature of 13° C. and after a rest of 24 hours the elasticity was redetermined with the same load as before.

Number of trial.	Alteration of resistance in terms of the divisions of the platino-iridium wire produced by 3 kilogs. at 13° C.	Alteration of resistance in terms of the divisions of the platino iridium wire at 100° C.	Alteration of resistance after cooling again to 13° C.
1	51·85	51·13	51·25
2	51·85	51·25	51·33
3	51·88	50·25	51·27
4	51·98	49·80	51·18
5	52·02	51·65	51·08
6	51·80	53·15	..
7	51·82	52·00	..
8	51·73	51·75	..
9	52·38	50·33	..
10	52·08	50·22	..
11	51·85	50·75	..
12	51·92	51·15	..
13	..	52·43	..
14	..	52·43	..
Mean	51·93	51·30	51·23
Probable error per cent.	·06	·16	·03

It will be observed that even at 100° C. the probable error does not amount to ·2 per cent., and, moreover, besides the set of readings recorded in the experiment two others of a similar kind were made afterwards, giving mean values of 51·21 and 51·26, and a total mean of 51·28 as the alteration of resistance caused by a load of 3 kilogs. at 100° C., as against 51·23, the alteration produced on cooling again to 13° C.

Experiment XLVI.

The same wire was again heated to 100° C. and tested with loads of 1, 2, and 3 kilogs., and afterwards, having cooled down to 16° C., the effects of the same loads were redetermined.

Number of kilogs. in the load.	Alteration of resistance produced by the load at 100° C.	Average alteration per kilog. at 100° C.	Alteration of resistance after cooling for 24 hours to 16° C.	Average alteration per kilog. at 16° C.
1	16·89	16·89	17·40	17·40
2	35·82	17·91	34·33	17·17
3	54·99	18·33	51·62	17·21

Experiment XLVII.

The same wire was treated with a load of 6 kilogs. in exactly the same way as it had been treated when 3 kilogs. were employed, and afterwards tested with loads from 1 to 6 kilogs.

Number of kilogs. in the load.	Alteration of resistance at 12° C.	Average alteration per kilog. at 12° C.	Alteration of resistance at 100° C.	Average alteration per kilog. at 100° C.	Alteration of resistance after cooling for 24 hours to 12° C.	Average alteration per kilog. at 12° C.
1	16·96	16·96	18·60	18·60	17·42	17·42
2	33·81	16·92	37·25	18·63	35·53	17·77
3	51·97	17·32	55·92	18·57	52·50	17·50
4	69·98	17·50	75·02	18·76	70·80	17·70
5	86·03	17·21	92·84	18·57	88·45	17·69
6	102·84	17·13	111·95	18·66	106·10	17·68
Mean values	..	17·17	..	18·63	..	17·63

Experiment XLVIII.

The same wire after having been heated to 100° C., and kept at this temperature for several hours with a load of 6 kilogs. on it, was allowed to cool, and in 24 hours the alteration of resistance produced by the cooling observed.

Similar processes were employed with loads of 5 kilogs. and of 3 kilogs.

Number of kilogs. left on the wire when cooling.	Superior alteration of resistance of stretched wire produced by the cooling from 100° C. to 13° C. – signifies superior decrease of resistance of stretched wire on cooling.
6	−8·8
5	−8·0
3	−6·6

Experiment XLIX.

An annealed copper wire, 7 feet in length and ·095 centim. in diameter, treated in a manner similar to that in which the last iron wire had been treated, and tested at 12° C. and 100° C. with a load of 5 kilogs.

Temperature in degrees Centigrade.	Alteration of resistance in terms of the divisions of the platino-iridium wire produced by 5 kilogs.
12°	42·13
100	45·34
12	44·31
100	45·14

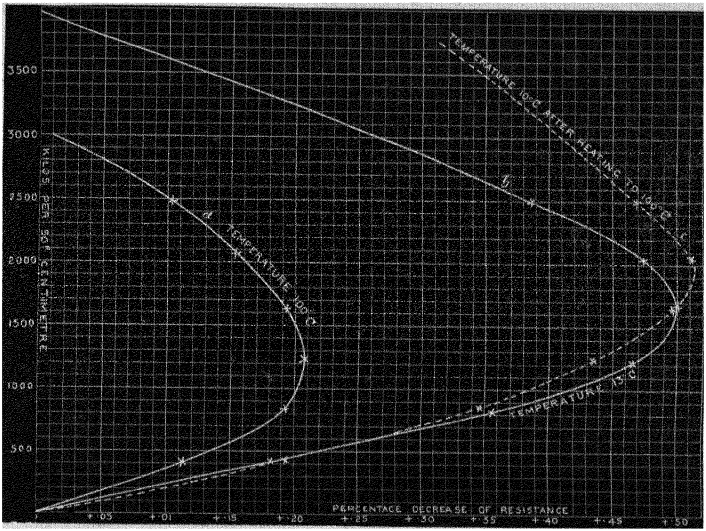
Experiment L.

A piece of the same nickel wire, which had been previously broken by testing for alteration of resistance at the ordinary temperature of the room, was annealed again and adjusted in the air chamber in the manner previously described. A load of 18 kilogs. was then suffered to remain on the wire for several minutes, and after its removal the nickel was allowed to rest unloaded for 24 hours. The wire was now tested with loads up to 12 kilogs. at the temperature of 13° C., then after heating to 100° C., and again, 24 hours afterwards, when cooled to 10° C.

Load in kilogs	Alteration of resistance at 13° C in terms of the divisions of the platino-iridium wire. – signifies decrease of resistance of stretched wire	Alteration of resistance at 100° C.	Alteration of resistance after cooling to 10° C
2	– 58·3	–36·1	– 60·6
4	–114·0	–62·8	–110·0
6	–148·4	–68·2	–137·7
8	–159·1	–62·5	–155·9
10	–151·3	–49·6	–160·7
12	–123·8	–33·8	–149·4

The effect of change of temperature on the temporary decrease of resistance produced by loading is also shown in Table XXIII.

TABLE XXIII.—Curves showing the temporary alteration of electrical conductivity of nickel produced by longitudinal traction at temperatures of 10° C., 13° C., and 100° C.



Remarks on the last experiments and on Table XXIII.

Experiment XLV. teaches us that though the alteration of resistance of the iron when loaded with 3 kilogs. is greater at 13° C. *before heating to* 100° C. than at 100° C.; yet, *after cooling again to* 13° C., the alteration is very slightly *less* than it had been at 100° C., and, on the whole, there is a *permanent* decrease of the temporary alteration of resistance produced by the loading. From Experiment XLVI. we learn that the second heating to 100° C. still further increases the difference of the effect of loading with 3 kilogs. at the lower and the higher temperatures; the alteration at the *higher* temperature being now more than 6·5 per cent. *greater* than at the *lower* temperature; but when the load employed is only 1 kilog., the alteration seems to be greater at the lower temperature than at the higher. Experiment XLVII. shows that when the wire was treated with 6 kilogs. there was a permanent increase of elasticity produced by the loading, and it should at the same time be noted that 6 kilogs. when first put on the wire, *barely produced a permanent increase of length*—an increase certainly not amounting to more than $\frac{1}{10}$th per cent. By comparing also columns two and six we can see that the slight annealing caused by raising to 100° C., and cooling again has diminished the elasticity, and therefore we have evidence here that *very slight extension* permanently increases the elasticity of iron; whilst in Part I. we have seen that moderate permanent extension decreases, and excessive permanent extension increases the elasticity of iron. It is evident, therefore, that we have three critical points in iron wire, and that the elasticity is first increased to a maximum, then decreased to a minimum, and finally begins to increase again as the permanent extension is gradually increased from exceedingly small amounts to the breaking point of the wire.

Again, the alteration of resistance produced by *all* the loads is now *greater* at 100° than at 12° by amounts varying from about 5 to 6 per cent.

Experiment XLVIII. also shows that when loads from 6 to 3 kilogs. are left permanently on the wire when cooling from 100° C. to the temperature of the room, the alteration of resistance decreases as we decrease the load, and bears out the previous observations that the elasticity is less at the higher than at the lower temperatures.

In Experiment XLIX. we have evidence that the slight annealing caused by raising the temperature of the copper to 100° C. permanently decreases the elasticity, and that the alteration of resistance at 100° C. is for the load employed about 2 per cent. greater than at 12° C. It is, however, when we come to Experiment L. that the most noticeable changes are seen to be produced, and when we consider the results recorded in this experiment and in Table XXII. we are led to the conclusion that the temporary alteration of susceptibility to change of resistance from changes of stress effected by raising to 100° C. is, with nickel, as remarkable as we have seen this susceptibility itself to be. The curves are drawn on the same scale as the curves for nickel at 22° C., and it will be observed that not only is the maximum diminution of resistance lessened by raising to 100° C. to an amount which is *less than one-half* of the maximum diminution at the lower temperature, but that also the load at which this maximum diminution occurs is much less; and it may well be that at a sufficiently high temperature the decrease of resistance which is observed to be produced by moderate loads at the lower temperatures would be changed to an increase. In fact, I am inclined very strongly to believe that there exists with all metals a critical temperature below which temporary stress will produce temporary decrease of resistance, and that above this temperature there is an opposite effect caused by the stress.

The value of "YOUNG's modulus" was determined for this piece of nickel by the method of static extension, in a manner to be presently described, and was equal to 2480×10^6 at the temperature of 16° C. and to 2280×10^6 at the temperature of 100° C. It will be observed that the maximum decrease of resistance as shown in Table XXIII., is appreciably greater than the maximum decrease observed with the other specimens,[*] and still further shows how the amount of this decrease depends upon the elasticity.

As regards the question whether the thermo-electric properties of the metals, as affected by temporary stress, be related to the susceptibilities of alteration of resistance from change of temperature in the same way as the corresponding qualities seem to be where permanent strains are concerned, it may be said to remain at present open, and only to be decided by further experiments on the effects of stress on the thermo-electric qualities of iron;[*] but as far as copper and nickel are concerned, the above question seems to be answered in the affirmative.

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[*] These two specimens were received at different times from Messrs. JOHNSON, MATTHEY, & Co., and the latter shows still more remarkably than the former the large elasticity which can be obtained from nickel even in a well-annealed condition.

THE EFFECT OF SLIGHT MECHANICAL STRAIN AND OF THE STRAIN CAUSED BY RAISING IRON TO 100° C. AND AFTERWARDS COOLING, ON THE TORSIONAL RIGIDITY OF THE METAL.

The above experiments had shown such an astonishing influence to be produced on the longitudinal elasticity of annealed iron by merely raising the metal to 100° C. and then cooling, that it seemed advisable to test whether or not a similar effect would be produced on the torsional rigidity of iron by a like cause, and if so, whether we can imitate the strain resulting from heating and afterwards cooling by mechanical means.

Experiment LI.

A piece of annealed iron wire was vibrated at a temperature of 13° C., and the time of a single vibration, as determined by counting the vibrations for five minutes, was found to be 1·000 second. The wire was then heated in an air chamber to 100° C., and after having been maintained at this temperature for one hour was suffered to cool, and the time of vibration found to be 0·989 second 12 hours after cooling.

Experiment LII.

A second piece of the same wire, when suspended ready for vibrating, was heated slightly by passing the flame of a BUNSEN burner rather quickly up and down it several times, the vibrator being at the same time supported, so as to take off stress from the wire. The time of vibration before heating was 1·154 second, and in five minutes, 35 minutes, and 245 minutes after cooling was 1·147, 1·142, and 1·136 second respectively.

The wire was then heated to redness and cooled, when the time of vibration after five minutes was 1·156 second, and after 22 hours became 1·143 second.

Experiment LIII.

A third piece of the same wire was vibrated after different slight amounts of permanent extension had been produced; the length before stretching was 82·0 centims., and the time of vibration 1·621 second. After slightly stretching, so as to increase the length to 82·2 centims. and 84·3 centims., the time of vibration became respectively 1·614 second and 1·714 second.

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[*] This point will be fully considered in Part IV.

Experiment LIV.

A piece of annealed pianoforte steel was vibrated after different slight amounts of permanent extension.

Length of wire, in centims.	Time of vibration, in seconds.
89 00	984
89·05	·976
89 10	968
89 20	968

These experiments* speak for themselves, and prove that the torsional rigidity is affected in a precisely similar manner to the longitudinal elasticity by raising to 100° C., and then cooling, and moreover that the strain produced by slight mechanical traction acts in a similar manner on both iron and steel to the strain produced by tempering. We see also how very quickly the increase of elasticity is changed into a decrease when the extent of strain is widened either by heat or by mechanical means. Evidently then there are for iron three critical points as regards its torsional rigidity as well as regards its longitudinal elasticity—very slight strain increasing, moderate strain decreasing, and excessive strain again increasing both these physical properties. Further, it would be interesting to determine whether cold would not produce the opposite effect on the elasticity to heat, and it seems highly probable that cooling below the temperature of the room will permanently decrease the elasticity of iron; this point, however, I hope to be in a position shortly to decide.

FURTHER DISCUSSION OF WERTHEIM'S EXPERIMENTS ON ELASTICITY.†

We have seen that the temporary alteration of resistance produced by any load is permanently decreased in the case of annealed iron wire by merely raising the temperature of the metal to 100° C. Now WERTHEIM'S experiments seem at first sight to prove that the elasticity of iron and steel is greater at 100° C. than at the ordinary temperature of the room; but if M. WERTHEIM had examined the elasticity after the wire tested at the higher temperature had again cooled down to the lower one, he would have found that this *apparent temporary* increase of elasticity was *really a permanent* one, and if the wire had been tested several times, first at the higher and then at the lower temperature, he would have also found, provided sufficient rest after cooling had been allowed, that the elasticity of both iron and steel is *temporarily diminished* by raising the temperature to 100° C.

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* All these experiments were repeated several times with different specimens of iron, but invariably with the same result as regards the nature of the change.

† Ann. de Chimie et de Phys., 3^{me} série, 1844.

From WERTHEIM'S researches* we gather that in the case of iron and steel there is the following increase of elasticity between 15° to 20° C. and 100° C.

TABLE XXIV.

Metal.	Increase per cent. of elasticity between 15° to 20° and 100° C.
Annealed iron .	4 90
Annealed iron wire .	6 91
Annealed cast steel .	2 79
Annealed English steel wire .	23 20
Steel tempered blue .	5 18

The increase of elasticity of the steel wire seemed so remarkable that it was determined to retry WERTHEIM'S experiments by direct observations of extension, using the same scale and vernier as have been already described in Part I., and placing the wire and the comparison-wire in the same air chamber as had been used in measuring the alteration of resistance produced by loading at different temperatures. The length of wire between the clamp, which rested on a wooden support on the top of the chamber, and the vernier was 4 feet 4 inches, and of this length 2 inches or rather more would be, roughly speaking, at the temperature of the room when the rest of the wire was heated by steam to 100° C. We must therefore multiply any observed effect on the elasticity caused by raising the temperature of the wire to 100° C. by a number which is about 4 per cent. of the apparent alteration. The air chamber rested (fig. 18) on a piece of wood, A A, 8 inches long, 6 inches broad, and ½ inch thick, and this in turn on two stout bricks, B B, which were supported by a wooden table, D D. The wire to be examined, X, and the comparison-wire, Y, were, as usual, secured at one end of each to the same clamp, C, which rested on a piece of hard wood, E E, placed at the top of the chamber. The wires, X and Y, passed through two small holes made in E E, and also through a hole in the table, and to their lower ends were attached, in the one case, a scale-pan, and in the other a cross-bar of wood, H H, carrying constant equal weights. Exactly the same precautions were taken as have been already fully described in Part I., and, as in experiments on the alteration of resistance produced by loading, the air chamber was well covered with baize and paper.

Under these circumstances it was possible to maintain a very constant temperature of 100° C., and even had the temperature varied slightly it will be seen that no appreciable error would have been introduced, as the variation would have altered the lengths of X and Y to the same extent.

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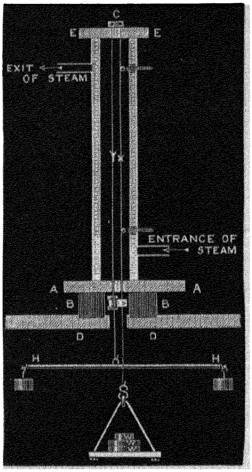
* Ann. de Chimie et de Phys., 3me série, 1844, p. 431.

Fig. 18.



Experiment LV.

An annealed pianoforte-steel wire, ·085 centim. in diameter, was loaded for some hours with a weight of 10 kilogs.; this was then removed and a rest of 24 hours allowed, when a weight of 4 kilogs. was left permanently on the wire and the elasticity determined at the ordinary temperature of the room, and at 100° C., by putting on and taking off a load of 6 kilogs., the recovery being quite perfect for the load at both the higher and the lower temperatures.[*]

Number of trial.	Alteration of length produced by 6 kilogs. at 12° C. in half-millims.	Alteration of length produced by 6 kilogs. at 100° C.	Alteration of length produced by 6 kilogs. after cooling to 12° C.
1	1·48	1·49	1·45
2	1·50	1·49	1·46
3	1·51	1·50	1·44
4	1·51	1·48	1·44
5	1·51	1·48	1·46
6	. .	1·48	. .
Mean	1·502	1·487	1·450
Probable percentage of error . . .	·40	·27	·30

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[*] At the higher temperature after the *first* loading, which was not taken into consideration in estimating the elasticity.

Experiment LVI.

The same iron wire as had been used for Experiment LV. was loaded for some time with 8 kilogs.; 4 kilogs. were then removed and the wire allowed to rest for 24 hours, when the effect on the length of the wire produced by putting on and taking off 2 kilogs. was found to amount to an alteration of ·940 half-millim. at 100° C. After the action of the steam had been stopped for two hours and the temperature was about 20° C., the alteration produced by the same load amounted to ·985 half-millim., and after 20 hours rest at the temperature of 13° C. was found to be ·945 half-millim. The wire was now tested with 4 kilogs., when the alteration of length produced by this new load was 1 910 half-millim. at 13° C., and after heating for some hours to 100° C., 1·911 half-millim. The source of heat having been removed and the wire allowed to cool slowly for 20 hours, the alteration caused by the same load was 1·880 half-millim. at 13° C

Experiment LVII.

An annealed nickel wire, ·09213 centim. in diameter, was loaded for several hours with a weight of 18 kilogs.; 10 kilogs. were then permanently left on the wire for 24 hours, and the alteration of length effected by a load of 8 kilogs. at a temperature of 16° C amounted to 1·250 half-millim. The wire was heated to 100° C., and the alteration was now found to amount to 1·360 half-millim. After cooling down for the next five hours to a temperature of about 20° C. the alteration became 1·350 half-millim., and after three days at a temperature of 15° C. proved to be 1·250 half-millim.

Experiment LVIII.

The same copper wire as had been used in Experiment XLIX. was loaded for some hours with a weight of 11 kilogs.; 7 kilogs. were then allowed to remain on permanently and the wire suffered to rest for two days. It was then raised to the temperature of 100° C. with the weight of 11 kilogs. on, and cooled again to the temperature of the room. These operations were repeated each four times during a space of four days, and on the last of these the alteration of length produced by a load of 4 kilogs. was found to be 1·374 half-millim. at 100° C.; on cooling to 15° C. the alteration caused by the same load proved to be 1·310 half-millim., and this, or nearly the same alteration, had been found to be produced by the load at the temperature of the room, which temperature varied from 15° C. to 16° C. on each of the previous days.

The following table exhibits the difference of the alteration of electrical resistance and of length at the temperature of the room and at 100° C.

TABLE XXV.

Metal.	Percentage temporary alteration of elasticity caused by raising the temperature from 15° C. to 100° C + signifies increase of elasticity produced by rise of temperature		Percentage increase of alteration of resistance caused by raising the temperature from 15° C to 100° C. − signifies decrease of alteration.
	WERTHEIM.	TOMLINSON.	
English steel wire .	+23·20	−2·58	
Iron wire	+ 6·91	−1·64	+ 5·70
Copper .	− 6·59	−4·74	+ 1·87
Nickel .		−8·41	−50·00

Remarks on Experiments LV.–LVIII., and on Table XXV.

It appears from the last experiments that there is really a permanent increase of elasticity produced in annealed iron and steel by merely raising the temperature to 100° C.; and what is still more remarkable, there was in the case of one specimen of iron wire, which was so ductile as to lengthen by 24 per cent. before breaking,* such a large loss of ductility that its maximum permanent elongation *barely reached* 13 *per cent. after it had been heated to* 100° *C. and allowed to cool again.* As the last discovery was made when experimenting on the effect of permanent extension on the susceptibility of the resistance of iron to change of temperature, it was thought at first that the passage of the current used in determining the electrical resistance might have some influence on the ductility, but on heating three other pieces of the same wire to 100° C., and afterwards allowing them to cool without permitting any current to flow through them, almost exactly the same change was observed; and yet in all these cases the rate of cooling was slow; so much so, indeed, that with the last specimen employed, in which special precautions had been taken to surround the small air chamber in which the wire was heated with several coatings of baize, the chamber was sensibly warm to the touch 12 hours after the cooling had commenced. How far other kinds of annealed wrought-iron might show a change of ductility from like cause I know not, but it seems evident that changes comparatively enormous can be produced in the elasticity and ductility of this metal by small alterations of temperature.

It has also been shown by Experiment XXXIX. that the electrical conductivity of annealed steel is increased by heating slightly and then cooling quickly, and it will be proved in Parts III. and IV. that there is a correspondingly large effect on the magnetic inductive capacity and on the thermo-electric properties of steel produced by the same process. It would appear, therefore, that researches of this kind might lend valuable aid in investigations on the liability of wrought-iron axles to fracture produced by sudden changes of the temperature of the air.

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* This wire I received through the kindness of Mr. J. T. BOTTOMLEY, and was especially prepared for experiments on magnetic induction carried on in the physical laboratory of Sir W. THOMSON.

Again, it will be observed from Experiment LVI. that shortly after the iron has been heated and then cooled there is less elasticity than when a considerable rest has been allowed; and in fact we have in this case exactly the same kind of restitution of elasticity in iron as we have seen takes place after the wire has experienced mechanical extension. With nickel the increase of elasticity produced by rest after cooling is still more remarkable. Equally remarkable also is the temporary change of elasticity produced in nickel; and a comparison of the loss of elasticity produced by raising the temperature to 100° C. with the change of susceptibility to alteration of resistance from change of stress as shown in Table XXIII. affords a still further proof that stress and strain act in the contrary direction as far as electrical conductivity is concerned.

It will also be noticed in Table XXV. that the alteration of susceptibility to change of electrical resistance from change of stress is greater in iron in proportion to the alteration of elasticity when the temperature is raised to 100° C., and that the converse is the case with copper, so that there is a greater or less alteration of *specific resistance* for the loaded wire than for the unloaded, caused by rise of temperature to 100° C., according as the metal is iron or copper; and we have seen that a similar state of things occurs where the strain is that left after the removal of the stress.

THE ALTERATION OF ELECTRICAL CONDUCTIVITY PRODUCED BY MAGNETIZATION.

History of the subject and description of apparatus.

This subject has, in the case of iron, received the attention of several observers, who have in some instances differed not only as regards the amount but also as regards the nature of the change produced by magnetization on the electrical resistance. Sir W. THOMSON, in 1856,[*] was I believe the first to show that magnetization affected the electrical conductivity of iron and steel, longitudinal magnetization causing in these metals increase, and transverse magnetization decrease of resistance, and this with steel was found to be the case whether the metal was hard or soft. Shortly afterwards experimenting on nickel, THOMSON found[†] with this metal also an alteration of resistance similar in kind to that of iron, but greater in amount for the same magnetizing force. With brass, on the contrary, he failed to detect any change whatever.

Subsequently BEETZ,[‡] CHWOLSON,[§] and myself[||] pursued similar investigations.

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[*] Phil. Trans., "Electrodynamic Qualities of Metals," Part IV.

[†] Proc. Roy. Soc., vol. viii., 1857.

[‡] POGG. Ann., vol. cxxviii., p. 202 (1866).

[§] CARL's Rep., vol. xiii., p. 232 (1877).

[||] Proc. Roy. Soc., June 17, 1875, vol. xxiii., p. 533.

BEETZ and CHWOLSON both confirmed the results of THOMSON as far as longitudinal magnetization was concerned ; but with the former of the two first observers an experiment where transverse magnetization was employed, ended in giving purely negative results. The values obtained in my own experiments differed considerably from those of THOMSON as regards amount and in the case of hard steel also in nature. Moreover, it appeared that the circular magnetization which ensues when a current is passing through an iron wire caused an increase of resistance, and as we might perhaps expect circular magnetization to cause a similar effect, as regards nature, on the conductivity to transverse magnetization,[*] these results were not in accordance with those of THOMSON. Unfortunately I had not at the time read Sir W. THOMSON'S paper, and investigations made in later years convinced me that the observations recorded in my "Preliminary Notice" were not reliable, partly because alteration of resistance from change of temperature had not been sufficiently guarded against, and partly because of a "PELTIER effect," which I have since found would, with the large battery-power employed in the circuit of the "WHEATSTONE'S bridge" arrangement, vitiate the results. To my astonishment, AUERBACH, three years afterwards, published an essay[†] "On the Passage of the Galvanic Current through Iron," in which my own observations both as regards the magnitude and nature of the changes produced in iron and hard and soft steel by longitudinal magnetization were apparently fully confirmed. Moreover, it appeared that he also agreed with me that circular magnetization caused increase of resistence in iron ; and ingeniously reasoning that, this being the case, feeble longitudinal magnetization should decrease the resistance of iron, brought forward a series of experiments which seemed fully to bear out his views. Under these circumstances it seemed to be advisable to retry some of my old experiments on iron and steel *rods*, and further to extend the enquiry to *wires* of iron and steel.

As several of the instruments used in this part of the investigation have also been employed in several experiments yet to be described in the other parts of this paper, it seems advisable to give here some description of them, as well as certain data which will be required for converting the measurements taken into absolute units.

The tangent galvanometer.

This was of the usual HELMHOLTZ-GAUGAIN pattern, made by ELLIOTT Bros., where there are two stout copper rings for measuring currents where it is desirable that the resistance of the galvanometer should be small, and three pairs of coils of finer wire for other purposes. In these investigations the former only were employed, and therefore we may consider the resistance of the galvanometer itself to be neglectable.

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[*] This, however, is not, I believe, the case.

[†] Phil. Mag., July, 1879, vol. viii., p. 1. Translated from the original essay (Leipzig, 1878), communicated by the author.

The short needle was as usual provided with an aluminium index, and by means of this, readings on a graduated circle traversed by the ends of the index could be depended upon to within $\frac{2}{10}$ths of 1°. The needle was suspended by a platinum wire $\frac{1}{1000}$th of an inch in diameter and 12 inches in length, the upper end of the wire being secured to a torsion-head provided with a vernier, which moved over a graduated circle; by means of the vernier a torsion amounting to six minutes could be measured. The fine platinum wire hung in the axis of a brass tube which was provided near its lower extremity with a small glass window, so that the illuminated image of a vertical wire focussed by a lens could be received on a small mirror and reflected back on a scale placed at a distance of 1000 of its own divisions from the mirror. The mirror was attached to a piece of stout brass wire which was connected at its lower extremity with a needle, and at its upper extremity was clamped to the lower end of the platinum wire.

The effect of the torsion on the deflection was determined by a series of careful observations made by turning the torsion-head through different angles, first in one direction and then in the opposite, and noting the corresponding deflection of the needle. Thus, in one experiment, the torsion-head having been turned through 100 degrees from left to right, there was produced a deflection of 57° 6′, and when the head was turned from right to left, the deflection was 56° 46′ on the opposite side of the zero point; the mean of the two deflections of the needle is 57°, and therefore the force of 1° of torsion of the suspension wire would $=\frac{\text{sine } 57^\circ \times \text{H}}{(100-57)}=\cdot 0195\times\text{H}$; where H is the horizontal force of the earth's magnetic action on the needle.

In order to be able to determine in absolute units the value of the deflections of the needle, a new DANIELL's cell of large size was charged with a saturated solution of sulphate of copper and a semi-saturated solution of sulphate of zinc. This cell was allowed to rest for two hours after having been charged, and was then short-circuited for half an hour. After this time the cell was connected up with the thick copper wires of the galvanometer, and the deflection of the needle produced, observed by means of the mirror and scale, when external resistances of 0 ohm, 10 ohms and 20 ohms were successively introduced. The deflections were reversed in each case by reversing the battery-current and the mean values of the two deflections in the different directions were for 0 ohm, 324 divisions of the scale; for 10 ohms, 110 divisions; and for 20 ohms, 30 divisions; these deflections corresponding to 8° 59′, 3° 8′, and 0° 52′ respectively. By comparing the deflection with 0 ohm and 10 ohms in circuit, the internal resistance of the battery, together with the resistance of the connecting wires, was found to be 1·055 ohm, and by comparing the deflections with 10 ohms and 20 ohms in circuit, this same resistance was calculated to be 1·058 ohm; therefore, 1·057 ohm was assumed to be the true resistance. Again, taking the electromotive force of the DANIELL thus charged to be 1·12 volt, or $1\cdot 12\times 10^8$ electromagnetic units, and the resistance of 1 ohm to be 10^9 of these units, the formula for converting the readings of the galvanometer into electromagnetic units was determined to be

$$\gamma = {\cdot}3158\ (\tan\phi + {\cdot}00839\phi\sec\phi),$$

where ϕ equals the deflection of the needle in degrees, and γ is the value of the current strength in absolute measure.

The magnetizing coils.

During a considerable part of these investigations two magnetizing coils were employed, of which descriptions are now given.

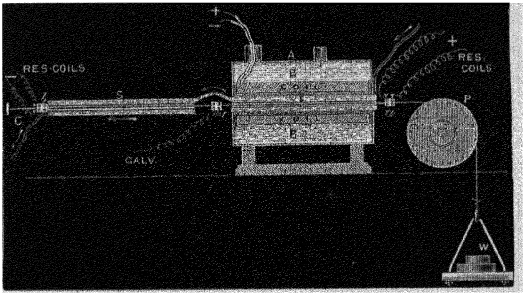
The larger of the two coils, which will be designated as the coil A, was made of about 1000 feet of cotton-covered copper wire, $\frac{1}{20}$th of an inch in diameter; the whole piece was divided into seven equal portions, and these, after having been well soaked in melted paraffin wax, were placed side by side and bound together by tape wound spirally along the whole length of the compound strand thus formed. This strand, after a further soaking in paraffin, was wound on a stout glass tube having an internal diameter of 1 inch and a length of 13 inches. The glass tube was provided at its two extremities with discs of hard wood $\frac{1}{4}$ inch thick and 6 inches in diameter, and surrounding the whole of the coil was a zinc cylinder concentric with the glass tube, which being closed at its two ends by discs of the same metal served when filled with water to keep the coil cool (fig. 19). The ends of each of the seven portions into which the whole wire was divided were connected with separate terminal screws so that the seven coils could be used either in "series" or in "multiple arc." When arranged in "series," in which form they were employed in this particular branch of the enquiry, the total resistance of the coils at 15° C. was 4·464 ohms. The compound strand was distributed along a length of 30 centims. in layers of five deep, so that the total number of turns amounted to 2100. The inner diameter of the coil was 3 centims. and the outer 7 centims.; therefore the magnetizing force at the centre and the two extremities would be respectively $4\pi \times c \times 70 \times {\cdot}986$ and $4\pi \times c \times 70 \times {\cdot}498$ absolute units, where c denotes the current strength, and the average force throughout the whole length would be $4\pi \times c \times 70 \times {\cdot}742$.

The smaller of the two coils, the coil B, was constructed as follows:—A thin tube of polished brass, with a slit running throughout its entire length, $1\frac{1}{4}$ inch internal diameter and $4\frac{1}{2}$ inches long, was covered with vulcanised caoutchouc to a depth of $\frac{1}{16}$th of an inch, and on this was wound 3 lbs. of cotton-covered copper wire, $\frac{1}{20}$th of an inch in diameter, followed by 3 lbs. of wire, $\frac{1}{10}$th of an inch in diameter. Inside the first tube was placed a second of similar kind, 1 inch in internal diameter, and connected with rings of ebonite with the first at the two ends. The second tube was concentric with the first, and of the same length, so that between the inner tube and the outer there was interposed a layer of air nearly $\frac{1}{8}$th of an inch in thickness. This arrangement was employed to prevent the heat from the magnetizing coil reaching any wire or rod placed inside. The wire forming the coil had a resistance of 1·782 ohm at the temperature of 15° C., and was distributed in 814 turns in a length of 9·7 centims. The inner diameter of the coil was 3 centims. and the outer diameter 9·0 centims., so that the magnetizing force at the centre would be $4\pi \times c \times 90{\cdot}4 \times {\cdot}851$, and that of the two extremities $4\pi \times c \times 90{\cdot}6 \times {\cdot}478$ absolute units; consequently the average force throughout the coil would be $4\pi \times c \times 90{\cdot}4 \times {\cdot}664$. As with coil A, the whole of the wire was well soaked in paraffin wax before winding, and the insulation between one layer and another was, by determining the resistance before and after winding, ascertained to be quite perfect in both A and B.

THE EFFECT OF TEMPORARY STRESS ON THE ALTERATION OF ELECTRICAL RESISTANCE PRODUCED BY MAGNETISM.*

In order to test the effect of temporary and permanent extension on the change of electrical resistance produced in iron wire by magnetism, the following arrangement was adopted:—The wire, which had a total length of from 4 to 5 feet, was firmly secured at one end (fig. 19, C) to a strong upright, and was then slipped through two glass tubes of small diameter about 13 inches in length; the glass tubes were placed inside two copper vessels, S S, of the same length as themselves. These vessels each consisted of two concentric cylinders connected by soldering at the two extremities by copper rings and provided with small pipes, so that cold water could be kept running through the space between the cylinders. The outer of the cylinders was 1 inch nearly in external diameter, and the inner ¼ inch in diameter; one of the copper vessels was placed inside the magnetizing coil, A, whilst the other was supported on the table.

Fig. 19.



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* It should be mentioned here that in all these experiments the magnetizing coils were placed in a direction *perpendicular to the magnetic meridian.*

The wire passed over a large wooden pulley, P, and to the other end of it was attached a scale-pan for holding weights. The light clamps *a*, *b*, *c*, having been well secured to the wire, were connected up in the usual manner, as shown in the figure, with the galvanometer, battery and resistance-coils, the connecting wires in this case being for a distance of 2 feet of rather fine silk-covered copper, in order to avoid the strain which would probably have ensued from the comparatively heavy caoutchouc-covered wires ordinarily employed. The whole arrangement was then carefully covered over with baize, and water allowed to trickle from a small cistern through S, S for about an hour, the space B having been previously filled with water. After a sufficient time had elapsed to render the parts of the wire of the same temperature, the effects on the resistance of magnetism alone or magnetism combined with strain were determined. The tangent galvanometer was placed in the circuit of the coil A* for the purpose of measuring the current passing through the coil.

Experiment LIX.

An annealed iron wire, ·093 centim. in diameter. B.C. produced by one LECLANCHÉ. M.C.† by 10 GROVE cells with adjustable resistances in the external part of the circuit.

Number of kilogs. on the wire, scale pan weighing 2 kilogs. not included.	Tangent of deflection galvanometer = tan α.	Alteration of resistance in terms of the platinum-silver wire = d; one division = an alteration of ·00081 per cent.	$\frac{d}{\tan \alpha}$.
0	123	48·6	395
0	·249	65·0	261
8	249	40·0	
0	·105	40·0	381
4	·105	27·0	
6	·105	22 0	.

Experiment LX.

Another piece of the same wire was loaded and unloaded several times with a weight of 12 kilogs., and was afterwards allowed to rest for several days unloaded. The changes of resistance were then determined by noting the position of the light on the scale, the B.C. being produced by a large DANIELL's cell, the circuit of which, as well as that of the galvanometer, was kept closed as soon as the balance had been established. The effect of magnetization on the resistance was tested when loads of 0, 6, 10, and 12 kilogs. were on the wire. The M.C. was produced by five GROVE cells.

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* In the figure the manner in which the compound strand of the coil is arranged in "series" is not shown.

† The current in the circuit of the "WHEATSTONE's bridge" will be denoted by B.C. and that in the magnetizing circuit by M.C.

Tangent of the deflection of the needle of the tangent galvanometer.	Load in kilogs. on the wire when under magnetization, the scale pan weighing 2 kilogs. not included.	Alteration of resistance produced by magnetization in divisions of the galvanometer-scale, one division representing an alteration of ·0005 per cent.
247	0	27
	6	19
	10	25
	12	25

Both these experiments show that the increase of resistance which was produced by the longitudinal magnetization is lessened by temporary stress up to a certain limit of the latter, and several other experiments of a similar kind proved that after the diminution of alteration of resistance caused by magnetization had reached a maximum, further temporary stress began to reverse the first effect, sometimes only just before the "breaking-load" of the wire had been reached. In no case, however, was *diminution* of resistance caused by longitudinal magnetization for the highest stress which could be put upon the wire without breaking it,* and this, too, when strengths of current of very different degrees were tried.

Experiment LXI.

An annealed nickel wire, ·105 centim. in diameter, was arranged in the same manner as the iron wire in the last experiment; but the clamps *a* and *c* (fig. 19) were placed nearer together, and just inside the coil A, so that the whole of the nickel experimented on would be under the influence of the magnetizing force. In this case the coolers, S S, were dispensed with, and, instead, the wire to be tested was provided with a solenoid of fine silk-covered copper wire, wound in two layers on a glass tube of the same length as that of the nickel wire under examination, and of a diameter such that the latter could be easily slipped inside it. This solenoid served, when required, to give the relative amounts of magnetism imparted to the steel by the different magnetizing forces. The alterations of resistance produced by the magnetism were first determined, then the galvanometer having been disconnected from the "bridge" and joined up with the solenoid, the induced currents caused by the magnetization were measured by the

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*** This was rather unexpected; since JOULE has shown (Phil. Mag., 1847, vol. xxx., pp. 76, 225) that whilst iron free from stress is increased in length by longitudinal magnetization, yet when loaded beyond a certain limit its length is diminished by the same cause.**

"throw" of the needle, both with and without the B.C. flowing through the nickel. A preliminary set of observations had given the means of determining the amount of current which would be induced when the nickel was not in the coil, and, therefore, by subtraction the induced currents due to the nickel only could be determined. The glass tube and the fine silk-covered copper wire on it would have served to shield the nickel from any change of temperature likely to be caused by the magnetizing current; but as a further precaution, the solenoid was well wrapped up in paper so that it would just fit inside the coil A. The comparison-wire was also surrounded with glass and caoutchouc tubing, and, as with the iron, the whole arrangement was well covered over with baize.

The following values (Table XXVI.) of the alterations of resistance are the means of five or six trials with each of the various magnetizing forces employed.

TABLE XXVI.

M C in divisions of the scale of the tangent galvanometer α.	Throw of the galvanometer needle due to induction current caused by magnetization of the nickel when the load on the wire = 0 kilog. β.	Increase of resistance in terms of division of the platino iridium wire when the load on the wire = 0 kilog. γ.	$\frac{\gamma}{\alpha}$	$\frac{\gamma}{\beta}$.	Increase of resistance calculated from the formula $\gamma = a\,\alpha + b.\beta$ $a = 2992$ $b = 1514$.	Increase of resistance caused by magnetism with 2 kilogs on the wire.	Increase of resistance caused by magnetism with 6 kilogs. on the wire.	Increase of resistance caused by magnetism after the removal of the 6 kilogs. and after a rest of two days.
19	8 7	7 0	368	805	7 00	6 0	2 0	8 6
35	16 2	12 4	354	765	12 92	..	..	
53	25 6	19 1	360	746	19 73	.	..	
69	31 0	24 6	·356	·781	25 30	24 5	19 8	33 1
86	36 8	31 3	·364	·851	31 30	..		
159	56 4	57 0	358	1 011	56 10	53 0	45 0	80 0
279	81 5	95 8	343	1 175	95 80	99·1	85 2	131 0

It is evident from the last experiment that the increase of resistance which can be produced by magnetizing nickel wire longitudinally is diminished by temporary longitudinal stress not carried beyond a certain limit, provided the magnetizing force does not exceed a certain critical value depending upon the amount of stress applied. Thus we see that when the value of α is somewhere between 159 and 279, the load of 2 kilogs. begins to increase the alteration of resistance caused by the magnetization. Now, Sir W. THOMSON has proved*—and in the case of this particular wire I have been able to verify the fact†—that with nickel the magnetism induced by any magnetizing force is increased or diminished by stress according as the magnetizing force does or does not exceed a certain critical value. With iron, on the contrary, the induced magnetism is increased by stress, provided the magnetizing force does not reach a certain critical value, which however is very much less, other circumstances being the same, than is the case with nickel. It is possible, therefore, that if much smaller magnetizing forces had been employed in Experiment LIX., the increase of resistance caused by magnetism would be found to be heightened by the loads employed in that experiment.

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* "Electrodynamic Qualities of Metals"—Part VII., Phil. Trans., Part I., 1879.

† I shall have occasion to refer to this experiment in Part III. of my paper.

THE EFFECTS OF PERMANENT LONGITUDINAL EXTENSION OF TORSION AND OF TEMPERING ON THE ALTERATION OF ELECTRICAL RESISTANCE PRODUCED BY MAGNETIZING.

The last experiment shows that moderate permanent logitudinal strain largely increases in the case of nickel the susceptibility to alteration of resistance from longitudinal magnetization. A similar effect is produced on iron, but as the whole point will be more fully discussed in Part III., it will suffice here to state that several experiments made according to the above plan, and also others where the comparison-wire and the wire to be tested were placed together in the magnetizing coil, proved, undoubtedly, that moderate permanent strain *increases* the susceptibility, but that this increase, after reaching a maximum, begins to decline, so that in some cases after the wire had been broken by the stress applied, the susceptibility appeared to be less than it was before the wire had been subjected to any strain. The above-mentioned maximum point depends upon the amount of magnetizing force in a manner to be hereafter described.

The effect of permanent torsion up to a certain point was to *diminish* the susceptibility to alteration of resistance from longitudinal magnetization, and the amount of diminution was independent of the direction either of the M.C. or the B.C.

The following experiment shows that in hard steel longitudinal magnetization increases the electrical resistance.

Experiment LXII.

A piece of a steel knitting-needle, 7·62 centims. in length and ·23 centim. in diameter, was hardened by heating it to a bright red and then plunging it into cold water. The piece was connected up with another of similar dimensions and similarly prepared, and both having been well covered with caoutchouc and silk, were made to form two branches of a "WHEATSTONE'S bridge." The coil B was used to impart magnetism, and with a large LECLANCHÉ for the B.C., and four GROVE'S cells for the M.C., an *increase* of resistance represented by 30 divisions of the iridio-platinum wire was obtained. The brass clamps used to connect the pieces of steel with each other, and with the other branches of the "bridge," were so massive that even in this case their resistance is neglectable; and since 30 divisions of the iridio-platinum wire would show an alteration of resistance of ·010 per cent., we may assume that this last number represents approximately the extent to which the conductivity of the steel was diminished by the longitudinal magnetization.

Moreover, the steel under examination was well within the coil, the comparison-piece being, of course, outside, and also at right angles to the coil; and it was calculated that the increase per unit of resistance would for unit magnetizing force be $\frac{\cdot 00010 \times 9\,7}{814 \times 68 \times 4\,\pi \times c} = 69\cdot 8 \times 10^8$, since c, the strength of the magnetizing current here, approximately amounted to ·2 absolute unit.

It will be observed, by comparing this last result with the corresponding one in Experiment LXVIII., that the alteration of resistance produced by a given magnetizing force is very much less with the hardened steel than that caused by the same magnetizing force in a steel rod of the same diameter, but in the same condition as it was when received from the makers. Several other trials were made with the same piece of steel, in which smaller and smaller amounts of M.C. were employed, but in no case could any alteration of the nature of a *decrease* of resistance be observed.

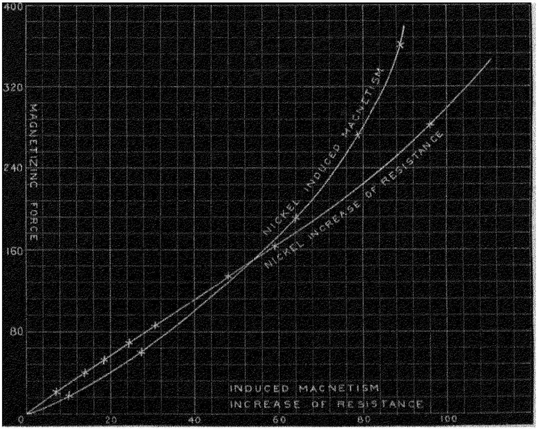
A similar experiment had been tried with pianoforte steel wire, hardened in the same way, and with various amounts of M.C., but the results were of the same nature, though much less in amount, as with the knitting-needle. It may be added that both the knitting-needle and the wire were made so hard that they were quite brittle, and with both there was a permanent as well as a temporary increase of resistance produced by the magnetization.

AN ATTEMPT TO DETERMINE RELATIONS BETWEEN THE ALTERATION OF ELECTRICAL RESISTANCE PRODUCED BY ANY MAGNETIZING FORCE, THE FORCE ITSELF, AND THE MAGNETISM INDUCED BY THE FORCE.

On consulting the fourth, fifth, and sixth columns of Table XXVI., it will be observed that the increase of resistance ensuing from magnetization depends not only upon the magnetism induced, but also upon the magnetizing force itself; and, in fact, we may say that if γ denotes the increase of resistance, whilst α and β represent the magnetizing forces and the magnetism induced respectively, $\gamma = a.\alpha + b.\beta$, where a, b are two constants. In the case of the nickel wire, α and β were measured in terms of the divisions of the scale of the tangent galvanometer, and of the scale of the THOMSON'S reflecting galvanometer respectively; whilst a and b were calculated from the observations made with $\alpha = 279$ and $\alpha = 86$. The agreement between the observed and calculated values of γ is good, and certainly quite equal to that between the different observations made with the same values of α. Columns 4 and 5 show clearly that the alteration of resistance depends in this case more on the value of the magnetizing force than on the magnetism induced, and from the fact that $\frac{\gamma}{\alpha}$ is nearly constant throughout, whereas $\frac{\gamma}{\beta}$ rapidly increases for the higher values of α, we are led to infer that the alteration of resistance would go on increasing as the magnetizing force increased, even when there might be no appreciable advance in the value of the magnetism induced.

A glance at the curves in Table XXVII. will serve to confirm the above view. These curves have their abscissæ representing both the induced magnetism and the alterations of resistance produced by the various magnetizing forces, which latter are measured by the ordinates of the curves. The induced magnetism[*] is represented on a scale of 1 millim. to one division of the scale of the THOMSON'S reflecting galvanometer, and the alteration of resistance on a scale of 1 millim. to one division of the platino-iridium wire. The ordinates are on a scale of 1 millim. to two divisions of the scale of the tangent galvanometer, and each of these latter divisions represents a current of ·00023 absolute unit. Each division of the iridio-platinum wire represents an alteration of resistance $=\cdot 000034$ per unit; if therefore we take the average of the first six values of $\frac{\gamma}{\alpha}$, namely, ·360, as representing the average alteration of resistance effected by a magnetizing current producing a deflection of one division on the scale of the tangent galvanometer, we find that the increase of resistance per unit produced in the nickel wire by unit magnetizing force $=\frac{\cdot 36\times\cdot 000034\times 30}{\cdot 00023\times 2100\times\cdot 75\times 4\pi}=8070\times 10^{-8}$.

TABLE XXVII.—Curves showing the increase of resistance and the amount of induced magnetism produced in nickel wire by different magnetizing forces.



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[*] There was no appreciable difference between the induced magnetism as determined with the B.C. flowing and that without.

Experiment LXIII.

An annealed iron wire, ·094 centim. in diameter, was arranged with the same precautions and in the same manner as the nickel wire in the last experiment; but as it was found difficult to make observations in the ordinary way in consequence of the VILLARI'S "shock-currents" being very pronounced, the B.C., for which one GROVE'S cell was employed, was kept flowing until the wire and the comparison-wire had assumed a sufficiently stable resistance-ratio which was very nearly equal to unity. The alteration of resistance produced by various magnetizing forces was measured by the deflection of the image of the illuminated wire on the scale, and the mode of taking the readings and the nature of the corrections to be applied for the direct action of the magnetizing coil on the galvanometer are described in Experiments LXIX. and LXXI.

The following table contains the results of this experiment:—

TABLE XXVIII.

M.C. in divisions of the scale of the tangent galvanometer. α	Throw of the galvanometer needle due to the induction current caused by the magnetization of the iron. B.C flowing β.	Throw of the galvanometer needle due to the induction current caused by the magnetization of the iron. B.C not flowing β'.	Increase of resistance in terms of divisions of the galvanometer scale. γ.	$\frac{\gamma}{\alpha}$.	$\frac{\gamma}{\beta}$.	$\frac{\beta}{\beta'}$.	Increase of resistance calculated from the formula $\gamma=a.\alpha+b.\beta$ $a=\cdot0397$ $b=\cdot106$.
21	20	17	1·9	091	·095	1·18	2·9
43	56	42	5·0	·116	090	1·33	7·7
82	107	71	13·1	·160	1·220	1·51	14·6
130	134	91	19·3	·148	1·440	1·47	19·4
203	153	103	23·9	118	1·562	1·49	23·9
383	200	..	36·9	·096	1·845		36·4
519	220	172	43·9	·085	2·000	1·28	43·9

Table XXVIII. shows that with iron as with nickel the increase of resistance can be represented by the formula $\gamma=a.\alpha+b.\beta$,* and, here again therefore, it seems as if the alteration of resistance would go on increasing as the magnetizing force increased even when there would be no perceptible increase of induced magnetism. The cases of iron and nickel, however, differ considerably in one respect, namely, that whereas with the latter metal the magnetizing force played the more important part in altering the resistance, with the former, for moderate values of the M.C. the greater part of the alteration is effected by the induced magnetism.

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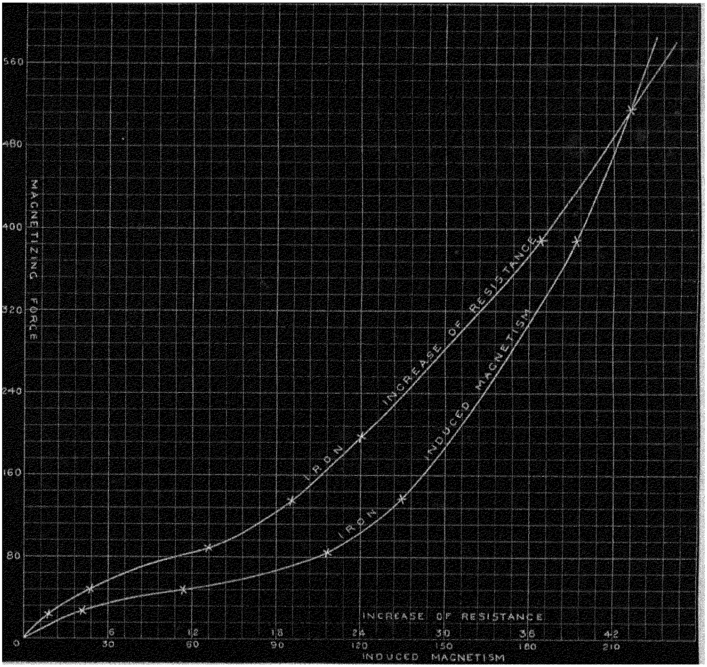
* The discrepancies between the observed and calculated values of γ for the first three magnetizing forces is, I believe, almost if not entirely due to the fact that, unfortunately, before any exact measurements had been made with the lower values of α, the full magnetizing force had been employed. This would not perhaps have affected the result, as far as the agreement between observation and calculation is concerned, had the induced magnetism been measured at the same time as the alteration of resistance, but this was not the case.

Table XXIX.—Curves showing the increase of resistance and the amount of induced magnetism produced in iron wire by different magnetizing forces.



The curves in Table XXIX. showing the amount of magnetism induced and of the alteration of resistance are constructed on the same lines as those in Table XXVIII., except that in consequence of the alteration of resistance being much less and the induced magnetism greater with iron than with nickel, the alteration is represented on a scale of 1 millim. to ·3 division of the scale and the induced magnetism on a scale of 1 millim. to $1\frac{1}{2}$ divisions of the scale. Each division of the scale represents an alteration of resistance amounting to ·000023 per unit, and if we take the mean of the third and fourth values* of $\frac{\gamma}{\alpha}$, which may perhaps be assumed to represent approximately the average alteration of resistance for a given moderate magnetizing force, we find that with this wire the increase per unit of resistance produced by an absolute electromagnetic unit of magnetizing force would be 2335×10^{-8}.

It is desirable to draw attention also to the fact that with both iron and nickel there is no change of resistance of the nature of a *decrease* produced by magnetization, but that starting with a current from one GROVE's cell through a total external resistance of about 15 ohms† and with a magnetizing force not greater than 26 times the earth's horizontal magnetic force at the place, we find a continuous *increase* of resistance as the magnetizing force is increased.

THE EFFECT OF ALTERING THE STRENGTH OF THE B.C. ON THE CHANGE OF RESISTANCE PRODUCED BY ANY MAGNETIZING FORCE.

The results recorded in the last two experiments are so far at variance with AUERBACH'S views already alluded to, that it seemed advisable to still further test these views by altering the strength of the B.C. whilst that of the M.C. is maintained constant.

Experiment LXIV.

A piece of the same annealed nickel wire as that used in Experiment LXI. was tested with one GROVE'S cell and a total external resistance of 15 ohms for the M.C., and with from one to three GROVE'S cells with no external resistance save that offered by the "bridge" and its connexions for the B.C.

Deflection of the needle of the tangent galvanometer, showing the strength of the B.C.	Increase of resistance caused by the magnetization in terms of the divisions of the iridio-platinum wire.
12°	10·1
15	10 3
17½	10 9

The numbers given in the second column are the means of several observations, and agree very fairly with each other, the difference between them being within the errors of observation, and what difference there is would show that we have a slightly *greater* increase of resistance for large values of the B.C. than for small ones.

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* The first two values of $\frac{\gamma}{\alpha}$ are not included, for the reason previously mentioned.

† That is, 10 ohms in addition to the resistance of the coil and its connecting wires.

Experiment LXV.

A piece of the same iron as had been used in Experiment LXII. was tested with one GROVE'S cell and no external resistance save that of the magnetizing coil for the M.C., and three GROVE'S cells with adjustable external resistance for the B.C.

B.C. in terms of the divisions of the scale of the tangent galvanometer.	Increase per unit of resistance M.C. = 109 divisions of the scale of the tangent galvanometer.
130	000300
940	000316
1521	000353

In this experiment, which was conducted in the same manner as Experiment LXII., there is evidently a *greater* alteration produced by magnetism when the B.C. has a high value than when it has a low one, and the differences between the different values in the second column are certainly larger than could be attributed to errors of observation. Now in Experiment LXVIII. it will be shown that with unannealed steel, and in Experiment LXI. it has been shown with annealed nickel, that there is little or no difference in the amount of alteration of resistance effected by magnetism when the B.C. is made to vary in amount; and the reason is apparent, for with the nickel and the steel there was no* appreciable difference between the induced currents caused by the magnetization of these metals when the B.C. was flowing and when it was not, whereas, if we turn to the second, third, and seventh columns of Table XXVIII., we see that there is a very appreciable difference in the case of the annealed iron wire, between the induced currents with and without the B.C.

All these experiments are in direct contradiction to those of AUERBACH, but yet it was thought fit to try others with annealed pianoforte-steel wire.

Experiment LXVI.

A piece of annealed pianoforte-steel wire, 10 centims. long and ·085 centim. in diameter, was tested with various battery-power from one to four GROVE'S cells for the B.C., and battery-power varying from one GROVE'S cell with a resistance of 10 ohms in the external circuit besides the resistance of the coil A, to seven GROVE'S cells with no external resistance save that of A for the M.C.

In no case was a *diminution* of resistance produced by magnetization. With one GROVE'S cell for the B.C., and seven GROVE'S cells for the M.C., an increase of resistance of ·0585 per cent. was observed: lower values of the M.C. gave smaller and smaller results as the M.C. diminished. The alteration of resistance produced by unit magnetizing force was estimated in the usual manner to be 1500×10^{-8} per unit.

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* That is, no difference of such an amount as to make it seem worth while at the time to record it.

Experiment LXVII.

A strand of four pieces of the same length and of the same steel as that used in the last experiment was tested with one GROVE's cell in the B.C., and seven in the M.C., and the alteration of resistance, which, however, could not be accurately measured, was certainly not *greater* than that of the single wire when the same battery-power was employed for both the M.C. and the B.C. The change of resistance of the compound strand under the above-mentioned conditions was measured at ·040 per cent. with a probable error of 25 per cent.

THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF A BAR OF STEEL.

As my earliest investigations recorded in the previously-mentioned "Preliminary Notice" had seemed to show that the alteration of the electrical resistance of iron and steel which can be produced by magnetization is very much greater, for the same amount of magnetizing force, when the metals are in the form of comparatively thick rods than in that of wires, the following experiment was made.

Experiment LXVIII.

A steel knitting-needle, taken in the ordinary condition, 23 centims. in length and ·233 centim. in diameter, was provided with two copper terminals, 14 centims. in length and ·410 centim. in diameter, holes having been bored 1½ centim. in depth at one end of each terminal so as to admit the ends of the needle. A similar and similarly-furnished needle served as the comparison-piece, and the two were connected with each other and with the other parts of the "bridge" in the same manner as the wires in the last experiments had been. The clamps used, however, were more massive and the whole of the steel to be magnetized was well within the coil A, whilst the comparison-piece was outside the coil and at right angles to it. The amount of magnetism induced was measured as before, but the resistance introduced into the circuit of the galvanometer employed for this purpose had to be made very considerably greater than was the case with the iron wire, in order that the "throw" of the needle might be reduced to the proper extent. The B.C. was produced by four GROVE's cells, and the M.C. by seven GROVE's cells, each with adjustable external resistance. The following are the results obtained :—

TABLE XXX.

M.C. in divisions of the scale of the tangent galvanometer $=a$.	"Throw" of the galvanometer needle due to induction current by magnetization of the steel $=\beta$.	Increase of resistance when B.C. = 19¼° of tangent galvanometer, the increase measured in divisions of platino-iridium wire.	Increase of resistance when B.C. = 29°.	Increase of resistance when B.C. = 38°.	Mean increase of resistance $=\gamma$.	Increase of resistance calculated from formula $\gamma' = \cdot 0386 \times a + \cdot 0588 \times \beta$.	$\frac{\beta}{a}$.	$\frac{\gamma}{\beta}$.
23	7	1·35		.	1·35	1·30	31	·193
108	32	6·48		. .	6·48	6 05	·30	·203
152	48	8 83	8 50	8 75	8·69	8·69	·32	·181
267	82		15·00		15·00	15·13	·31	·183
347	102	18 88	20·30	19 00	19 39	19·39	·30	·190

From this experiment we learn that the magnetism induced in the steel is, for the magnetizing forces employed, very nearly proportional to these latter, and also that the alteration of resistance is nearly proportional to the induced magnetism, and therefore to the magnetizing force. Still more closely can the alteration of resistance be calculated from the formula $y = a.\alpha + b.\beta$, where the constants a and b, given as ·0386 and ·0588, are determined from the alterations of resistance caused by values of the M.C. equal to 347 and 152 divisions of the scale of the tangent galvanometer.

Alteration of the strength of the B.C. seems to have little or no influence on the change of resistance produced by a given amount of magnetizing force, and therefore we may assume that the product of the mean values of $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\beta}$ will fairly represent the alteration which would be wrought by a current in the magnetizing coil which would suffice to deflect the needle of the tangent galvanometer through one division of the scale.

This product $= \cdot 0589$, and since a division of the iridio-platinum wire corresponds to an increase per unit of the resistance of the steel $= \cdot 000032$,* and, since also one division of the scale of the tangent galvanometer represents a current of ·00023, whilst the average magnetizing force due to unit current would in the present instance be $\frac{2100 \times \cdot 82 \times 4\pi}{30}$, we see that the maximum increase of resistance obtained by the magnetization was ·062 per cent., and that the increase of resistance per unit for unit magnetizing force would be 1137×10^{-8}.

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* Of course correction is here, and in all similar cases, made for the resistance of the terminals, which, however, with these rods was very small.

THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF A BAR OF NICKEL.

Experiment LXIX.

A bar of nickel, 8·3 centims. long and ·70 centim. in diameter, was soldered to two stout copper terminals, whilst a similar bar, similarly provided, served as a comparison piece. The bars were arranged in the same manner as the steel bars in the last experiment, but the magnetizing coil B was used instead of A. Before placing the nickel in B it was covered with several layers of stout caoutchouc, and the comparison piece having been furnished in like manner, the usual precautions of well covering both bars were taken. The B.C. was furnished by two GROVE'S cells and the M C. by eight GROVE'S cells with adjustable external resistance. The circuit of the B.C. was kept closed, and the alteration caused by magnetization in the resistance of the bar was measured by the deflection of the image of the illuminated wire on the scale.*

As the resistances to be compared are in this case very small, it is advisable to show how far any measurements of alteration of resistance can be depended upon, and for this purpose the first set of readings with the smallest M.C are given :—

Total deflection caused by passing the M C. in terms of divisions of the galvanometer-scale + signifies increase of resistance.†	Number of trial.
+8·50	1
+8·25	2
+7·00	3
+6·25	4
+7·75	5
+8·50	6
+8·00	7
+9·00	8
+8·25	9
+7·75	10
+7·25	11
+7·9	Mean

---

* In this, and in every other instance in which such a mode of measuring alteration of resistance was adopted, the direct action of the magnetizing coil and of the included metal core on the galvanometer when the latter was not in circuit was always determined by a separate set of experiments, as though such action was small, it could never be entirely avoided. The distance (several yards) of the coil from the galvanometer was however such that no perceptible difference in the sensibility of the latter was introduced when the M.C. was closed. Similar remarks apply to the tangent galvanometer, and the readings given are in the case of both instruments always corrected for the above mentioned direct action. For the mode of taking the readings see Experiment LXXI. on Bismuth.

† Each number is calculated from three consecutive readings in the manner described in Experiment LXXI.

The deflection due to direct action of the electromagnetic solenoid and included core when the galvanometer was disconnected from the "bridge" $= +2{\cdot}9$.

Therefore the deflection due to alteration of resistance $= +5{\cdot}0$.

Immediately after taking the above readings the effect of altering by ·1 ohm the side of the "bridge" adjacent to the side containing the bar was found to be a deflection of 190 divisions on the scale, and since the ratio of the resistance of the bar and its copper terminals to that of the comparison piece was $\frac{12{\cdot}2}{10{\cdot}0}$, it was assumed that the increase of resistance amounted to $\frac{5{\cdot}0 \times {\cdot}1}{190 \times 12{\cdot}2}$ per unit. The deflection of the needle of the tangent galvanometer was in this case 4°, and a similar set of observations was made when the M.C. produced deflections of 6·25°, 8·25°, 10·75°, and 13°, with the results recorded below.

TABLE XXXI.

Deflection of tangent galvanometer $= \phi$.	$\text{Tan}\,\phi + {\cdot}0084\phi \sec \phi = c$.	Increase of resistance due to magnetization in terms of divisions of the galvanometer scale $= a$.	$\frac{a}{c}$.	$\frac{a}{c^2}$.
4°00	104	5 0	48 1	462
6 25	·163	9 8	60 1	369
8 25	215	17 0	79 1	368
10 75	284	24 0	84 5	298
13 00	343	32 5	94 4	275

It would seem that the increase of resistance produced by the magnetization varies in this case more nearly as c^2 than as c; if, however, we take the mean value of the numbers for $\frac{a}{c}$ we shall probably obtain a sufficiently close approximation to what can only be regarded as a rough measurement of the effect of magnetization on the resistance. This mean value is 73·2. The resistance of the nickel only was to that of the nickel and the connexions in the ratio of 3·8 to 12·2, and since the value of c must be multiplied by ·316 in order to obtain the value of the current in C.G.S. units, and since moreover a unit current would produce an average magnetizing force in this case of $\frac{814}{9{\cdot}7} \times {\cdot}7 \times 4\pi$, we see that the increase per unit of magnetizing force would on the whole be $\frac{73{\cdot}2 \times {\cdot}1 \times 9{\cdot}7}{{\cdot}316 \times 3{\cdot}8 \times 190 \times 814 \times {\cdot}7 \times 4\pi}$, or 4343×10^{-8}. The bar was cast, and used in the same state as sent by the makers;* when annealed the value given above would be considerably greater, as it was afterwards ascertained that annealing very largely increased the capacity for induction from moderate magnetizing forces.

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* For this bar and for the bars of cobalt and bismuth used in the next experiments I am indebted to Messrs. JOHNSON, MATTHEY, and Co. All the bars here mentioned were very nearly chemically pure.

THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF A BAR OF COBALT.

Experiment LXX.

A bar of cobalt, 8·5 centims. long and ·75 centim. in diameter, was provided with stout copper terminals, and balanced against a similar bar. The precautions taken and the mode of experimenting were exactly the same as with the nickel bar.

TABLE XXXII.

Deflection of tangent galvanometer $=\phi$.	Tan ϕ + ·0084ϕ sec ϕ $=c$.	Increase of resistance due to magnetization in terms of divisions of galvanometer scale $=a$.	$\frac{a}{c}$.	$\frac{a}{c^2}$.
4°·00	104	1·40	13·5	130
6·25	163	3·90	24·0	147
13·00	343	8·50	24·8	72

In this case the values of $\frac{a}{c}$ agree quite as well as $\frac{a}{c^2}$, and if we take the mean of the former, namely 21·1, we obtain an increase of resistance per unit attending magnetization by a unit force $=628 \times 10^{-8}$; a number, it will be noticed, only about one-seventh of that obtained in the case of the nickel. The cobalt was, like the nickel, unannealed, and annealing would have caused the effect of a moderate magnetizing force to be greater, though not so much greater as would be the case with nickel.

In both the nickel and cobalt bars there was a *permanent* increase of resistance produced by magnetization, the maximum alteration amounting in nickel to about ·125 per cent., and in cobalt to ·025 per cent.; so that the maximum permanent alteration of resistance which is caused by magnetization is much greater with nickel than cobalt.

With both bars the M.C. was found to produce very nearly the same alteration of resistance, whether passed in one direction or the other through the coil.

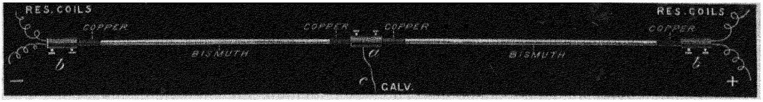
THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF A BAR OF BISMUTH.

Experiment LXXI.

As it seemed desirable to ascertain whether diamagnetic substances would have their resistances altered in the same direction as paramagnetic ones, some experiments were made with bars of bismuth, but as the magnetization which can be imparted by an electromagnetic solenoid to bismuth is very much smaller than is the case with iron, nickel, or cobalt, it is proper that rather fuller details should be given in describing the results obtained so that the trustworthiness of these results may be duly estimated.

A bar of bismuth, 25 centims. in length and ·330 centim. in diameter, was provided with copper terminals of the same diameter, and connected up with a similar bar to the "bridge," as shown in fig. 20. In this figure the relative dimensions of the bismuth bars and their copper terminals, as well as the mode of establishing the "bridge," are sufficiently shown. To the binding screw, *a*, a silk-covered copper wire, *c*, was soldered to connect the bars with the galvanometer, whilst the two binding screws, *b b*, were provided with silk-covered copper wires leading to the poles of one GROVE's cell and to two resistance coils of 100 ohms each in the usual manner. The coil A was employed, and the bars were arranged in the same manner and with the same precautions as the iron wire used in Experiment LIX., except that now it was necessary to dispense with the glass tubes in order that the bismuth bars could be slipped into the copper vessels through which the water flowed. The bars were, however, well varnished and covered with tissue-paper so as to ensure thorough insulation. The B.C. was kept closed and the alteration of resistance observed in the same manner as with the nickel and cobalt bars. The M.C. was produced by 13 GROVE's cells, and the deflection of the needle of the tangent galvanometer was 22·5°. The readings given below are determined in each case as usual from three consecutive readings; thus, a b c being three consecutive readings the recorded number is $\frac{a+c+2b}{4}$.

Fig. 20.



Number of trial.	Deflection of galvanometer-needle in scale divisions. + signifies apparent increase of resistance on magnetization.
1	10·00+
2	10·00+
3	9·50+
4	10·25+
5	11·00+
6	10·50+
7	10·25+
8	10·75+
Mean	10·23+

The B.C. was now taken off and the direct action of the magnetizing coil on the galvanometer was found to produce a deflection of 8·00+. Therefore there seemed to be an increase of resistance caused by magnetization represented by 2·23 scale divisions. The B.C. was again put on, and by taking out and putting in ·1 ohm several times it was ascertained that this caused a movement of the image of the wire through 74·0 divisions. Moreover, when the M.C. was flowing, putting in or taking out ·1 ohm caused the same effect, and therefore the M.C. did not affect the sensibility of the galvanometer. In this case ·1 represented an alteration of resistance amounting to ·001 per unit, and therefore the increase of resistance from magnetization would be $\frac{2\,23 \times 001}{74}$ per unit=·000031 per unit and ·0031 per cent.

The B.C. was now reversed and the following observations taken:—

Number of trial.	Deflection of galvanometer-needle in scale-divisions + signifies apparent increase. – signifies apparent decrease.
1	7 50–
2	8 00–
3	8·00–
4	7 75–
5	7 75–
6	8 25–
Mean	7 88–

The B.C. was again taken off and the direct action of the coil on the galvanometer appeared to be now 9·5–.* Here, therefore, there would be an increase of resistance equal to that represented by 1·62 scale divisions, and the amount of increase would be ·0022 per cent. Again, the resistances of 100 ohms on two sides of the bridge were replaced by 10 ohms, so that now the arrangement having become more sensitive, a set of observations similar to the above gave an increase of resistance equal to 3·88 divisions of the scale, corresponding to an increase of ·0029 per cent. From the three sets of observations it was concluded that the electrical resistance of bismuth is *increased* by longitudinal magnetization by ·0027 per cent. for the amount of magnetizing force here employed. The increase per unit of resistance for a unit magnetizing force would be 21×10^{-8}.

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* For some reason the direct action of the magnetizing coil on the galvanometer was never quite the same for both directions of the M.C.

THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF ZINC FOIL.

Experiment LXXII.

A piece of commercial zinc foil, 14 inches in length and ·040 millim. in thickness, was wrapped lengthwise round a soft iron bar of circular section, ½ inch in diameter and 15 inches in length, which had previously been coated with two layers of brown paper. The width of the foil was such that when wound round the bar the edges just overlapped. The foil was secured in position by fine yet strong twine, and having been covered with two folds of brown paper the whole was placed centrally in the coil A. At the two ends the foil was cut so as to allow of these ends being clamped in the usual manner in brass blocks, and a strip of foil of similar dimensions served as the comparison-piece. The same mode of experimenting and the same precautions were taken as with the bismuth bar. Seven GROVE's cells were used for the M.C., and these produced a deflection of 15° of the needle of the tangent galvanometer. Several trials which accorded very fairly with each other showed a mean *increase* of resistance represented by 3·8 divisions of the scale. The M.C. was reversed, and again several trials showed an increase of resistance, the mean value of which was represented by 3·5 divisions. From the data obtained it was calculated that the increase of resistance amounted to ·0148 per cent. The iron core having been removed no appreciable change in the resistance of the zinc foil could be detected on passing the M.C.

THE EFFECT OF LONGITUDINAL MAGNETIZATION ON THE ELECTRICAL RESISTANCE OF COPPER WIRE.

Experiment LXXIII.

A piece of silk-covered copper wire, 12 feet in length and $\frac{1}{50}$th of an inch in diameter, was doubled backwards and forwards so as to form a bundle 1 foot in length. The whole was then well coated with shellac varnish, and when dry inserted into one of the copper coolers (fig. 19), which was placed in the coil A. A similar bundle served as the comparison-piece, and the two bundles were connected up in the usual manner with the bridge. One GROVE's cell was used for the B.C., and the deflection of the tangent galvanometer produced by the M.C. was 16°. The B.C. was kept on for 10 minutes, and whilst still on the following readings were taken:—

Number of trial.	Apparent alteration of resistance caused by magnetization. — signifies apparent decrease of resistance
1	6 0 —
2	4 5 —
3	6 5 —
4	6 5 —
5	6 5 —
6	7 0 —
7	7 0 —
8	6 5 —
9	7·5 —
Mean	6 44—

The B.C. was now taken off, and the direct action of the coil on the galvanometer was found to be 6·50. There would therefore, on the whole, appear to be an *increase* of resistance of 0·6. The B.C. was again put on, and a similar set of observations to the above produced when the the M.C. was reversed, an apparent increase of 7·80 divisions; whilst the direct action of the coil on the galvanometer caused a deflection of 8·00 in the same direction, so that now, on the whole, there would appear to be a *decrease* of resistance represented by ·20 division of the scale. The mean result of the two sets of observations would give a decrease of resistance represented by ·07 division of the scale. Now, in this case, the galvanometer had been made so sensitive by proper use of the adjusting magnet that an alteration of ·1 ohm on one of the two sides of the bridge, containing each 100 ohms, caused a deflection of 300 divisions of the scale; accordingly 1 division of the scale would represent an alteration of resistance amounting to $\frac{1}{300000}$ per unit, and ·07 an alteration of *less than one in four millions.* It is needless to say that this experiment shows that there is no reliable change of resistance to be detected even with the comparatively large magnetizing force employed in this case—a force which would be more than 480 times that of the earth's magnetic horizontal force at the place. The copper in this case was the ordinary copper wire usually employed for electrical purposes, but other experiments were made with chemically pure copper, and these all yielded results quite as negative as those just recorded. There is no doubt that the electrical resistance of copper is altered by magnetization; but in order to detect such alteration we should, in all probability, require the aid of a very powerful electromagnet* and a galvanometer in a very sensitive condition. Such change of resistance must be exceedingly small even with the most powerful magnetizing force that we can at the present time bring to bear—much smaller than would seem to follow from Stewart and Schuster's experiments, some of the details of which will now be discussed.

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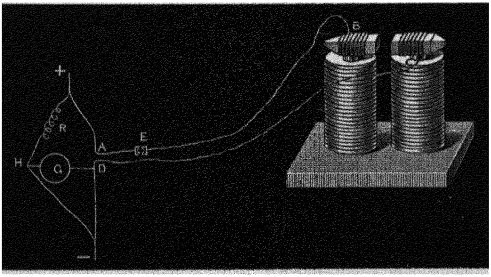
* One might, perhaps, by adopting the same plan as that used with the zinc foil, succeed in obtaining evidence of alteration of resistance in the case of copper foil.

Discussion of Stewart's and Schuster's Experiments on the Alteration of the Electrical Resistance of Copper by Magnetization.*

Stewart and Schuster, in a preliminary notice, bring forward the results of certain experiments which in their opinion seemed to prove that the electrical resistance of copper wire is altered by magnetization ; but a glance at their mode of operating serves to show that the effects observed by them cannot be relied upon. In fig. 21 (copied from Phil. Mag., 1874) A B C D is a caoutchouc-covered copper wire several yards in length wound round the armatures of an electromagnet, and R is another resistance against which the copper wire is approximately balanced. The alteration of resistance was observed from the "*throw*" of the needle of the galvanometer G, caused by closing the B.C. by means of the contact-breaker E, first when the electromagnet was actuated by six Grove's cells and then without any current in the M.C., or *vice-versâ*. The galvanometer was in such a position with reference to the electromagnet that the latter produced very little direct effect on the former, and since they obtained *momentarily* an increase of potential at D compared with the potential at H when the M.C. was passing, they inferred that the magnetization imparted by the electromagnet to the wire decreased the resistance of the copper.

Fig. 21.



These experimenters, however, seem to have entirely overlooked the fact that the powerful magnetization imparted by the magnet to the armatures would alter the capability of the latter to receive fresh magnetism. Now, when the B.C. is closed at E, the current in the coil of wire wrapped round the armatures would induce magnetism in the latter, and this would in turn send an *induced current through the caoutchouc-covered wire in the opposite direction to that of the original ;* if, then, the capability of receiving fresh magnetism is *lessened* in the armatures by previous powerful magnetization this induced current would be *lessened*, and thus, if their mode of experimenting was adopted, the resistance of the wire would *seem* to be decreased. The following experiment was tried with a view of testing the effect of previous magnetization on the capability of receiving fresh magnetism in the case of the armature of an electromagnet.

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* Phil. Mag., May, 1874.

Experiment LXXIV.

A bar of soft iron, 12 inches long $\frac{1}{2}$ inch broad and $\frac{1}{2}$ inch thick, was placed across the cores of a small electromagnet, and separated from them by a piece of tissue paper. The cores of the electromagnet were 1 inch in diameter and 5 inches in length, and were wound round with the cotton-covered copper wire $\frac{1}{16}$th of an inch in diameter, and having a resistance of nearly 1 ohm. A battery of six GROVE's cells was employed with the electromagnet and in order to test the change of susceptibility to magnetization, some 200 turns of rather fine silk-covered copper wire were made round one end of the bar of soft iron and distributed over a length of 4 inches. The coil thus formed was connected with the galvanometer, and was further insulated from the iron core by two layers of paper. The coil was then placed inside the coil B, which itself could be placed in the circuit of one large DANIELL's cell by means of a mercury-cup. The current induced by the magnetization imparted by the coil B to the soft iron bar was measured by the "throw" of the galvanometer-needle produced when the circuit of the DANIELL's cell was closed, first without exciting the electromagnet and then with this magnet in action. The electromagnet was at such a distance from the galvanometer that no error of importance caused by the direct action of the former would be introduced. The following experiments were then made :—

Condition of electromagnet. "On" signifies magnet excited, "off" not excited.	Deflection of the image of the illuminated wire on the scale.	Difference effected in the deflection by the action of the electromagnet. + signifies increase, — signifies decrease produced by exciting the electromagnet.	Number of trial.
Off	227		1
	226		2
	227	..	3
On	270	43+	1
	223		2
	186	41—	3
	186		4
	186	.	5
Off	212	26—	1
	202	16—	2
	202		3
On	233	31+	1
	187	15—	2
	187	.	3
Off	212	25—	1
	200		2
	198	11—	3
	198		4

The current through the electromagnet was then reversed.

Condition of electromagnet.	Deflection of the image of the wire.	Difference effected in the deflection by the action of the electromagnet.	Number of trial.
On	198		1
	192		2
	192	.	3
Off	250	58—	1
	212	20—	2
	212	20—	3
On	204	8—	1
	202	10—	2
	202	10—	3

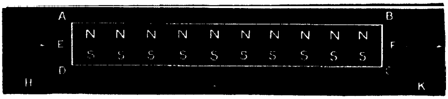
It will be noticed that for both directions of the M.C. the induced current is less with the electromagnet excited than when not.

Several phenomena connected with the experiments in the paper above alluded to can be explained by referring the apparent alteration of resistance to the permanent, sub-permanent, or temporary alteration of the magnetic susceptibility of the soft iron armatures employed; and, on the whole, when we consider that, as stated by them, the electromagnet had no apparent effect to make a piece of the wire set either axially or equatorially, we must regard their results with great suspicion.*

I myself seven years ago tried the effect of an electromagnet on a copper wire coiled several times round a flat piece of hard wood, and placed between the poles of a powerful electromagnet actuated by 12 GROVE's cells, but in no case could I detect the slightest real alteration of resistance. The galvanometer then employed was only able to detect an alteration of 1 in 50,000 of the resistance, and I look forward with some interest to renewed experiments in the same direction with the much more sensitive instrument which I have at present.

THE EFFECT OF ANNULAR MAGNETIZATION ON ELECTRICAL RESISTANCE.

Fig. 22.



Let A B C D, fig. 22, be a thin slice of iron (in the plane of the paper), through which an electric current is passing in the direction of the arrows. The molecules at the upper part, A B, will tend to take up positions such that their axes are perpendicular to the plane of the paper and with their north ends above it. The magnetization imparted to the molecules will diminish from A B to the axis E F, where it will be zero, and below this axis the molecules will be impelled to place their south poles above the plane of the paper. If now H K be an independent current passing below A B C D, it will tend to reverse the magnetism of the molecules at C D and strengthen that of the molecules at A B; but since the lower molecules are nearer to H K than the upper ones, the total effect would be a partial diminution of the annular magnetism imparted by the current flowing through A B C D. If, on the contrary, the current below the wire flows from K to H, the annular magnetism would on the whole be increased.

Now, some years ago, when making attempts to discover whether the resistance of an iron wire could be altered by passing a current above or below the wire, I was led to believe that such was the case. Among several other experiments, a knitting-needle provided with copper terminals was placed upon a strip of copper 16 inches long, 3 inches broad, and $\frac{1}{20}$th of an inch thick; the strip was well varnished, and the needle was laid upon it in such a position that the axis of the needle was coincident in direction with a line drawn from end to end of the strip and bisecting the two ends. On passing a current through the strip the resistance of the wire was altered in such a manner as would make it appear that circular magnetization *decreased* the resistance. I cannot, however, place reliance on these experiments, which at the time seemed conclusive, as I have in my recent attempts not been able to verify their results when sufficient precaution was taken to avoid change of resistance from change of temperature. The following are two experiments made with the above-mentioned object:—

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* It seemed to be the more desirable to test these results, as AUERBACH twice alludes to them in his paper (Phil. Mag., July, 1879, pp. 15–17), and should they not be correct, as I cannot help feeling is probably the case, others might be misled, owing to the well-deserved reputation of these experimenters.

Experiment LXXV.

An annealed iron wire, 4 feet in length and ·085 centim. in diameter, was firmly bound by tape for a distance of 2 feet with a caoutchouc-covered copper wire $\frac{1}{16}$th of an inch in thickness. The two parts of the wire were arranged as usual to form two sides of a "WHEATSTONE'S bridge"; and whilst the B.C. was varied in different trials to very different extents and the current through the copper wire was increased from almost 0° to 50° of the tangent galvanometer, *no trace whatever* could in any case be detected of alteration of the resistance of the wire by passing a current through the copper wire, though the arrangement was sufficiently delicate at times to show an alteration of *one in one million.* At one time, indeed, it was suspected that there was an alteration, but this was afterwards traced to a slight direct action of the M.C. on the galvanometer.

Experiment LXXVI.

A strip of annealed iron foil, 8 inches in length 2 inches broad and $\frac{1}{80}$th of an inch thick, was placed upon a copper strip 12 inches long 3 inches broad and $\frac{1}{10}$th of an inch thick, the two being separated from each other by two folds of a silk handkerchief. Another strip of iron of similar dimensions served as a comparison piece, and the two iron strips were arranged as usual in the "bridge." Though the currents through the strips of both iron and copper were altered to the same extent as with the iron wire, there was still no trace of alteration of resistance caused by the current in the copper strip. It is hardly necessary to say that such arrangements as these are not favourable for bringing out the effect sought, inasmuch as the distances of the copper wire and copper strip conveying the current were both greater than the thickness of the iron itself;* but, on the other hand, if we make the iron thicker we diminish the sensitiveness of the arrangement, and we certainly cannot well diminish the distance of the copper from the iron without laying ourselves open to error from changes of resistance caused by heating.

Several attempts were now made to ascertain whether variation of the B.C. itself would cause any variation in the resistance of iron wire, of which the next experiment will furnish a sample.

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* With the wire twice as great and with the strip nearly equal.

Experiment LXXVII.

An annealed iron wire, 14 inches long and ·85 millim. in diameter, was balanced against a platinum wire $\frac{1}{30}$th of an inch in diameter, and of nearly equal resistance. The B.C. was varied by interposing different amounts of resistance in the external circuit of one GROVE's cell, and this could be done very quickly by a suitable arrangement of mercury cups. Resistance coils of 10 ohms each were used in the two branches of the bridge, where 100 ohms resistances were generally employed.[*] As it was found impossible to allow the current to flow for even two or three seconds without unduly heating the wire, and as VILLARI's "shock-currents" would cause the resistance of the iron to appear greater than it should be when the B.C. was closed, and less than it should be when the B.C. was opened, the following plan was adopted:—The B.C. was, at intervals of 30 seconds, closed and *immediately* afterwards opened; and in this way it was found possible to obtain the value of the resistance-ratio of the iron and platinum so nearly, that on moving the sliding-piece to points on the iridio-platinum wire 20 millims. above or below the supposed balancing-point, a deflection of several divisions could be obtained in one direction or the other. As the contrary "shock-currents" produced by closing and immediately opening the B.C. would not quite neutralise each other's impulsive effects on the galvanometer-needle, it is obvious that the effects on the galvanometer, due simply to the fact of the sliding-piece being equal distances above or below the true balancing-point, would not produce equal deflections, but that, by taking the mean of these deflections, the true point might by easy calculation be determined. As a sample of the mode of experimenting we will take the following case:—The true balancing-point *seemed* to be 55 millims. to the right of the zero of the iridio platinum wire, and 3 ohms were at this time in the circuit of the B.C.; on moving the sliding-piece to 75 a deflection, on closing and immediately afterwards opening the B.C., of five divisions to the left was obtained (a left deflection would here indicate that the sliding-piece should be moved to the left in order to get the true balance). The sliding-piece was now moved to 35, and a deflection of six divisions to the right was obtained on closing and opening the B.C. From this we learn that the true balancing-point would be $35+\frac{6\times 40}{11}=57$ nearly. The resistance of 3 ohms was then removed, and similar observations gave 58 as the balancing-point. Immediately afterwards, the 3 ohms resistance having been again introduced, the point appeared to be 56. The true balancing-point when 3 ohms were in was therefore assumed to be $\frac{57+56}{2}$, and when the 3 ohms were removed to be 58. In this way the following results were arrived at as the means of several trials :—

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[*] It may be perhaps as well to state here that when small resistances were being compared, 10 ohms instead of 100 ohms were generally employed, though this fact has not been always mentioned.

Resistance in the external circuit of B.C.	Position of equilibrium of the sliding-piece	Number proportional to the B.C.
10	56 7	1·0
3	55 8	2·8
0	55·7	11·0

Now when we bear in mind that one division of the iridio-platinum wire only represents an alteration of resistance of ·0033 per cent., it appears evident that alteration of the strength of the B.C. can have but very small effect on the resistance, and even if there is any alteration it is of such a nature as to show that circular magnetization produces *decrease*, not increase of resistance.

DISCUSSION OF AUERBACH'S EXPERIMENTS.*

The results recorded in the last few experiments are so completely at variance with my own former observations,† and with those of AUERBACH that it is very desirable to attempt to account for the discrepancies. I cannot help thinking that AUERBACH experimenting, as it would seem, in almost precisely the same manner as I did in 1875, may have been misled in the same way as I now believe myself to have been. We both employed copper terminals to our iron wires whose resistances were in some cases even greater than that of the wire itself, and therefore necessarily not very thick, and moreover balanced this compound wire of copper and iron against a wire of German-silver. Now when a current is passing through a wire compounded of two metals placed end to end, a "PELTIER effect" is produced such that an electromotive force is developed which sends a current in the opposite direction to the original; so that, if we attempt to find the resistance of the compound wire in the usual manner by closing the battery circuit and shortly afterwards that of the galvanometer, we shall obtain an apparent value for the resistance which will depend upon the battery-power employed, upon the length of time that the battery circuit has been closed, upon the medium surrounding the wire,‡ and upon the thermo-electric power of the two metals forming the compound wire whose resistance we wish to determine. Nor is the "PELTIER effect" necessarily confined to the two junctions, for since no wire can be made perfectly homogeneous throughout, there must be unequal heating wherever two dissimilar parts of the wire join, and therefore consequently there must be developed at each such junction an opposing electromotive force. Indeed, may not the large resistance of such alloys as German-silver and platinum-silver be in a great measure due to a similar unequal heating at the junctions of the molecules of the several metals forming the alloy: with this difference, however, that here there would be an equalization of temperature the rapidity of which would be more and more approached in the case of a compound circuit made up of several pieces, say of iron and copper, as the distances between the consecutive junctions of these pieces became less and less?

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* Phil. Mag., July, 1879.

† Ibid., June, 1875.

‡ That is whether this medium tends to preserve the inequality of temperature at the two junctions or not.

Further, besides the error likely to arise from the "PELTIER effect," we must expect to encounter another, inasmuch as the heat generated in a wire by the passage of a current by no means necessarily produces the same alteration of potential at the two junctions. This may be from two causes: either different rises of temperature may be produced because the terminal at one end carries off more heat than that at the other, or because the metals which appear to be identical at the two junctions are not so.

That the "PELTIER effect" does come largely into play sometimes, we can convince ourselves by passing a current from a single cell of DANIELL for five or ten seconds through a thermopile, and then, after disconnecting the cell from the pile, putting the latter in circuit of a galvanometer; in such case a very considerable deflection can be obtained with a delicate instrument, and, indeed, we can even make the warmth of the hand, pressed against the face of one thermopile, generate such a "PELTIER effect" in a second pile connected with the first, that on disconnecting the two from each other, and then connecting the second with a reflecting galvanometer, a deflection may be obtained which can be rendered visible at a considerable distance. The following experiments will show that such errors as those above-mentioned are by no means merely theoretical.

Experiment LXXVIII.

A silk-covered German-silver wire, $\frac{1}{30}$th of an inch in diameter, was soldered at its two extremities to two copper terminals, $\frac{1}{16}$th of an inch in diameter and 6 inches in length, the whole forming a fairly accurate resistance coil of half an ohm. The coil was put for 20 seconds in the circuit of one GROVE's cell, and then, after the cell had been disconnected, and when a further period of five seconds had elapsed, connected by a mercury cup with a galvanometer; a deflection of 200 divisions was obtained, whereas, previously, there had been no perceptible deflection. After a rest of five minutes there was only a deflection of some 20 divisions. The battery was now reversed, and after a connexion with the coil for 20 seconds, a deflection of 170 divisions was obtained in the opposite direction.

Experiment LXXIX.

A piece of annealed iron wire with copper terminals, both of similar dimensions to those of the wire and terminals in the last experiment, was treated in the same manner as the German-silver, and deflections of 50 and 25 divisions, both on the same side, were obtained. The wire and its terminals were in this case well wrapped up in paper, and, as in the previous experiment, there was no sensible deflection before the GROVE's cell was used. These experiments, which are only two out of several which were made with different pairs of metals, show that with the German-silver the electromotive force generated by the "PELTIER effect" was so far greater than that due to any other cause that the deflections were nearly the same on both sides; whereas with the iron the current produced by the unequal heating of the two junctions from other causes than the "PELTIER effect" predominated, and this was found to be the case with several specimens of iron and copper.

In my experiments of 1875 I balanced iron wires or iron rods against wires of other materials, and using rather powerful electromotors (from one to six GROVE's cells) proved, as I thought, that the electrical resistance of iron increases with the intensity of the current employed in the "bridge;" but in these later investigations, in which, having a much more delicate galvanometer, I could obtain a measure of the resistance of the substance within 1 in 50,000 with a battery-power one-tenth of the smallest then used, I have been unable to detect with certainty any such change.

As for the discrepancies which exist between my present and former observations on both soft iron and hard steel, I can only attribute them to errors caused by the magnetizing coil being too close to the iron or steel to allow of sufficient protection from errors caused by heat radiated or conducted from the former, and which might increase the resistance of the metal as a whole, or cause apparent increase or decrease by unequal change of potential at the junctions of the two copper terminals with the iron or steel. At any rate, using both steel and iron of the same qualities as used then, but adopting more perfect thermal insulation and a more accurate mode of experimenting, I have been unable to detect any such considerable increase of resistance in the case of soft iron or soft steel, or any decrease of resistance of hard steel, as I did then. Now AUERBACH, with some of his specimens of iron and steel wires, obtained apparent alterations of resistance of 1, 2 and even 3 per cent.—alterations of decrease or increase which would have, in the case of specimens of a similar nature used by myself, sent the reflected image of the illuminated wire flying off the scale, whereas, instead of this, I found nothing but variations of resistance which never, with wires of a similar diameter, reached even to ·1 per cent., and this, too, with magnetizing forces which must have equalled those employed by AUERBACH.

AUERBACH, again,* seems to concur with BEETZ and others that the mere mechanical pull connected with magnetizing would have caused an apparent decrease of resistance in Thomson's experiment on the effect of transverse magnetization on the electrical resistance of iron;* but when one considers the small effect of even a far larger stress than could have been produced by magnetization on the electrical resistance of iron, as shown in Table I., these objections must, I think, vanish; and further, with nickel whose resistance Thomson has proved to be altered similarly to iron,† the effect of mechanical stress is of *an opposite* nature to that produced by magnetization.

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* *Loc. cit.*, p. 151.

Remarks on the Nature of the Alteration of Resistance which is Produced by Magnetization.

It will be observed in Table XXXIII., in which are given the values of the increase of resistance produced by unit magnetizing force, that of all the metals examined annealed nickel is the most affected,‡ and that next in order come soft iron, soft steel, cobalt, and bismuth. Evidently the condition of the metal may largely affect the susceptibility to alteration of resistance, and from what we have previously learned the thickness may do so also, but in a direction opposite to that which was at first expected, namely, that thick wires would be less affected than thin ones when the same B.C. and M.C. were employed.

Table XXXIII.

Name of metals	Condition.	Diameter in millimetres.	Increase of resistance per unit. produced by unit magnetizing force.
Iron	Annealed	0·94	2335×10^{-8}
Steel	Annealed	0·85	1500×10^{-8}
Steel	Unannealed	2·33	1137×10^{-8}
Steel	Very hard	2·33	70×10^{-8}
Nickel	Annealed	1·05	8070×10^{-8}
Nickel	Unannealed	7·00	4343×10^{-8}
Cobalt	Unannealed	7·50	628×10^{-8}
Bismuth	Unannealed	3·30	21×10^{-8}

Had the nature of the change of resistance been the same for mechanical longitudinal stress as for longitudinal magnetization in the case of *all* metals, there is nothing in the actual *amount* of alteration that might not lead us to suppose that the change of resistance from the latter cause is due to mere rotation of the molecules, as molecules, without regard to the electric currents, which, according to Ampère's hypothesis, are constantly circulating round these molecules. But when we find that with nickel longitudinal mechanical stress, which must cause rotation of the molecules to a certain extent, but without magnetic polarity, actually, unless carried to a very great excess, produces decrease of resistance, we are probably right in conjecturing that the change of resistance resulting from magnetization is in a great measure due to the fact that the current used in the "bridge" is encountered by a set of molecular currents circulating all more or less in the same direction, and in planes more or less at right angles to the direction of the former current as the induced magnetism is greater or less.

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* Phil. Trans., 1856, p. 741.

† Proc. Roy. Soc., vol. viii, 1857.

‡ It is remarkable that the value of the "rotational coefficient" of nickel should also exceed that of the other magnetic metals.

RELATION BETWEEN THE "ROTATIONAL COEFFICIENT" OF METALS AND THE ALTERATION OF RESISTANCE PRODUCED BY MECHANICAL STRESS.

E. H. HALL has discovered that when a strip of metal along which a current is passing is placed between the poles of an electromagnet in such a position that the lines of magnetic force are perpendicular to the plane of the strip, an electromotive force is developed in a direction at right angles both to the plane of the strip and the lines of force, and that thus the current is deflected. This deflection varies in amount and also in direction with different substances, and Professor HALL has recently read before the British Association a paper[*] on this subject, in which he gives a table showing the extent and direction of the deflection produced in several metals. The extent of the deflection in any substance depends among other things upon a certain constant designated by Professor HALL as the "rotational coefficient." In this table the sign + or − is prefixed to the number representing the coefficient according as the current is deflected in the same direction in which the conductor itself tends to move, or the opposite. Below is given HALL's table, and appended to it the numbers representing the increase of specific resistance per unit temporary increase of length when this latter is produced by mechanical stress. A + sign prefixed to the numbers denotes an increase, and a − sign a decrease of specific resistance.

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[*] 'Nature,' Nov. 10, 1881. (Abstract of a note on the above subject read by Professor E. H. HALL at the meeting of the British Association at York.)

TABLE XXXIV.

Name of metal.	"Rotational coefficient."	Temporary alteration of specific resistance per unit produced by temporary increase of length per unit. + signifies increase of resistance on application of stress.
Iron	+ 78	+2·618
Zinc	+ 15	+2·113
Lead	..	+1·613
Tin	+ 0·2	+1·630
Brass	− 1·3	
Platinum	− 2·4	+2·239
Silver	− 8·6	+1·617
Copper	− 10·0	+1·005
Aluminium	− 50·0	−0·420
Nickel	−120·0	−8·860

Considering that HALL himself is doubtful about the order of the metals in the centre of the list, there seems to be a well-marked relation between the "rotational coefficient" and the alteration of specific resistance from temporary mechanical stress. This relationship is strikingly apparent in the case of the metals iron, zinc, copper, aluminium, and nickel, and there can be but little doubt that results of extreme interest might be obtained by observations of the effect of mechanical stress and strain on the "rotational coefficient."

THE RELATION BETWEEN ELECTRICAL RESISTANCE AND "VISCOSITY."

Whilst endeavouring to find a relation between the electrical resistances of substances and their other physical properties, I was struck with the failure which I experienced in finding any in the case of those properties which have been already examined, except one, and that is one which as yet I have not had time to examine with anything like the care which I hope at some future period to be able to bestow upon it, namely, that which has been called by Sir W. THOMSON, in the case of metals, their "viscosity." The experiments, however, which have been made show clearly that there is in all probability a very close relationship between molecular friction and electrical resistance. It is proposed to make extended observations of the diminution of amplitude of vibration of wires of considerable length whilst the oscillations of very small amplitudes are magnified by a mirror attached to the vibrator; so that small vibratory molecular displacement may be obtained, and further to examine the change of "viscosity" produced by change of temperature. It suffices, however, for the present to say that of the pure metals already examined, copper, silver, aluminium, zinc, and tin, the order of their "viscosity" is the same as that of their specific electrical resistance.

SUMMARY OF PART II.

1. The electrical resistances of iron, steel, platinum, German-silver, copper, platinum-silver, brass, zinc, silver, aluminium, tin, lead, and carbon are temporarily increased by temporary longitudinal stress, the amount of increase being nearly, but not quite, proportional to the stress.

2. The *specific* electrical resistances of all the above metals, except aluminium, is likewise temporarily increased by temporary longitudinal stress; with aluminium, however, the specific resistance is *decreased*. The total resistance and the specific resistance of nickel are both *decreased* by temporary longitudinal stress not exceeding a certain limit, whilst beyond this limit further increase of stress begins to produce increase of resistance. The alterations, both increase and decrease, are very considerably greater with this metal than those of any of the other substances examined.

3. The temporary alteration of specific resistance caused by stress is much less with the alloys German-silver, platinum-silver, and brass than with the several components of these alloys; this would suggest an apparent relation between the change of resistance caused by alteration of temperature and that due to mechanical stress; the former effect, however, is very much greater than the latter, if we regard the alterations of resistance attending the same amount of expansion in each case, and there is no doubt that the increase of resistance ensuing on rise of temperature is due almost entirely to other causes than mere expansion.

4. The elasticity of carbon rods varies considerably with different specimens, even from the same maker, and is nearly proportional in this case to the eighth power of the density. Thick rods have generally a less density than thin ones, and less elasticity.

5. The specific resistance of carbon also varies considerably with different specimens from the same maker, but there is no apparent relationship between specific resistance and elasticity.

6. The increase of resistance caused by longitudinal stress is with different specimens of carbon as with different specimens of other substances very nearly proportional to the amount of temporary elongation produced by the stress, and though with the exception of tin and lead, the total alteration of resistance resulting from a given amount of stress is less with the metals which have been examined than with carbon, this is not so with regard to the alteration of specific resistance.

7. Compression produces on the electrical resistance of substances an effect of a contrary nature to extension.

8. Stress applied in a direction transverse to that of the current produces both temporary and permanent alteration of resistance of a nature opposite to that resulting from longitudinal traction.

9. Stress applied equally in all directions diminishes the total and the specific resistance of most metals.

10. The alteration of the melting-point temperature of ice can be readily and accurately determined by observations of the change of resistance produced by fluid pressure on metal wires placed in the ice.

11. Experiments on the permanent alteration of resistance of metal wires produced by stress furnish valuable information respecting the "limit of elasticity" of metals.

12. There are two "critical points" in every metal at which sudden changes occur in the ratio of the permanent extension produced by any load and the load itself, when the latter is gradually and carefully increased. The first of these two points fixes the true "limit of elasticity," and the second the true "breaking-point" of the metal. With iron there are three, and perhaps more "critical points."

13. The "critical points" are evidently in most cases closely related to the moduli of elasticity.

14. The total resistance of most metals is permanently increased by permanent longitudinal extension, but with nickel the total resistance is permanently *decreased*, provided the extension does not pass a certain limit; beyond this limit further extension produces increase of resistance.

15. The rate at which a wire is "running down" under the influence of a load can be very advantageously studied by observing the permanent increase of resistance produced by the load.

16. If P be the "breaking-load" of a metal wire, and p be the load actually on the wire, the decrease per unit of the velocity of the increase of resistance is inversely proportional to $P-p$: so that the actual "breaking-load" of a wire can be calculated from observations of the rate of increase of resistance when a loaded wire is "running down."

17. The above-mentioned proportion holds good not only for one and the same metal but for different metals.

18. The result of experiments on the influence of permanent extension on the temporary alteration of resistance which can be produced by temporary longitudinal stress verifies the statement made in Part I. that "the elasticity of a wire is diminished by permanent extension not exceeding a certain limit, but beyond this limit increased." The effect of permanent extension on the alteration of resistance which can temporarily be produced in nickel by traction is very remarkable.

19. Permanent extension, hammering and torsion produce, even when carried to excess, very small changes in the *specific* electrical resistances of metals. Most metals have their specific resistances increased by strain caused by the above-mentioned processes, provided the strain does not exceed a certain limit: beyond this limit further strain decreases the specific resistance. In the case of iron and nickel, on the contrary, the specific resistance is at first *decreased* and afterwards increased.

20. The strain caused by heating annealed steel to a temperature slightly and very much in excess of that of the room produces effects on the specific resistance of the metal of a kind similar to those caused respectively by small and great mechanical strains.

21. The change of density which can be effected in metals by permanent extension, hammering, or torsion is small.

22. The amount of recovery of electrical conductibility which is produced by time in all metals which are in a state of strain varies considerably with the nature of the metal; with platinum-silver the amount of recovery in a given time is very small, and with German-silver comparatively very large.

23. The recovery of electrical conductivity is in all cases attended with increase of longitudinal and torsional elasticity.

24. Metals may be divided into two classes, as far as the influence of permanent strain on the susceptibility to temporary change of resistance from change of temperature is concerned. In one class the strained wire is most increased in resistance by rise of temperature up to a certain limit, whilst beyond this limit further strain diminishes the first effect. In the other class the converse takes place.

25. There is a close relationship between the thermo-electric properties of strained and unstrained metals and their susceptibility to change of resistance from change of temperature.

26. The elasticity of annealed iron or steel is not *temporarily* but *permanently* increased by raising the temperature of these metals to 100° C. The ductility of annealed iron may also be very considerably and permanently diminished by the same process. Mechanical strain influences the elasticity in the same manner as the strain caused by tempering, and we may say of both kinds of strain that in the case of iron and steel there are three "critical points"—very slight strain increasing, moderate strain diminishing, and excessive strain again increasing both the torsional and the longitudinal elasticity.

27. The temporary alteration of elasticity which is effected in the case of nickel by raising the temperature to 100° C. is very noticeable. Still more remarkable is the temporary alteration of susceptibility to change of resistance from change of stress which is produced by the same means.

28. The electrical resistances of annealed iron, annealed steel, very hard steel, nickel, cobalt, bismuth and zinc are all increased by longitudinal magnetization. The alteration of resistance produced by the magnetization of annealed nickel is very remarkable.

29. The amount of increase of resistance produced in iron and nickel by longitudinal magnetization depends not only upon the magnetism induced, but also upon the magnetizing force, in such a manner that increase of resistance will be produced by increasing the magnetizing force, even when the latter does not cause any appreciable increase of magnetism.

30. The increase of resistance which is produced by magnetization is probably not merely due to the rotation of the molecules of the magnetized substance as molecules, but to the electrical currents, which according to AMPÈRE's hypothesis are constantly circulating round the molecules.

31. The "circular" magnetization which is produced when a current flows through a wire of iron does not appreciably alter the electrical resistance of the wire.

32. The effects of temporary stress and of permanent strain on the alteration by magnetism of the resistance of an iron or nickel wire are of a similar nature to those on the alteration of the magnetic susceptibility of these metals.

33. There is a very striking relationship, both as regards amount and direction, between the alteration of specific resistance, which can be produced in a substance by longitudinal traction and the "rotational coefficient" of the substance.

34. There is evidently an intimate relationship between the "viscosity" of a metal and its specific electrical resistance.

II. *On the Specific Resistance of Mercury.*

By Lord RAYLEIGH, *F.R.S., Professor of Experimental Physics in the University of Cambridge, and Mrs.* H. SIDGWICK.

Received April 24—Read May 4, 1881.

OUR experiments on the determination of the British Association unit of electrical resistance in absolute measure are detailed in two memoirs communicated to the Society.[*] The conclusion to which they led us is that

$$1 \text{ B.A. unit} = \cdot 9865 \frac{\text{earth quadrant}}{\text{second}},$$

but this result differs considerably from that obtained by some other experimenters, the original Committee included. Although in the present state of the question it is not desirable that the B.A. unit should fall into disuse, there can be no question as to the importance of connecting it with the mercury unit introduced now more than twenty years ago by SIEMENS. It will then be possible, as recommended by the Paris Conference, to express our absolute measurements in terms of mercury, by stating what length of a column of mercury at 0° of 1 square millimetre section has a resistance of 1 ohm. Accordingly the experiments about to be described relate to the expression in terms of the B.A. unit of the resistances of known columns of mercury at 0°.

This investigation was the more necessary, as the principal authorities on the subject, Dr. WERNER SIEMENS and Dr. MATTHIESSEN, had obtained results differing by as much as ·8 per cent.

The earlier determinations of SIEMENS were vitiated by the assumption of an erroneous value (13·557) for the specific gravity of mercury, a constant which it is necessary to know in order to infer the mean section of a tube from the weight of contained mercury. The error, pointed out by MATTHIESSEN, was afterwards[†] admitted by SIEMENS, who gives as the corrected expression of the relation between the two units,

$$1 \text{ mercury unit} = \cdot 9536 \text{ B.A. unit.}$$

On the other hand, the independent measurements of the resistance of mercury by MATTHIESSEN and HOCKIN[‡] gave

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[*] **Proceedings, April 12, 1881; Phil. Trans., 1882, Part II.**

[†] **Phil. Mag., xxxi., 1866.**

[‡] **Reprint of British Association Reports, p. 114.**

1 mercury unit=·9619 B.A. unit,

the mercury unit being defined as the resistance at 0° of a column of mercury 1 metre long and 1 square millimetre in section.

Our own experiments lead us to a value not differing much from that of SIEMENS. We find

1 mercury unit=·95418 B.A. unit.

If we assume that the B.A. unit is ·98651 ohm (in accordance with our determination), we find

1 mercury unit=·94130 ohm,

the ohm being 10^9 C.G.S. The same result may be expressed in another way by saying that the ohm is the resistance of a column of mercury at 0°, 1 square millimetre in section, and 1062·4 millims. in length.

Through the kindness of Dr. C. W. SIEMENS we have had an opportunity of comparing with the B.A. units a standard mercury unit (No. 2513) issued by Messrs. SIEMENS and HALSKE. At the proper temperature (16°·7) we find that its resistance is

·95365 B.A. unit,

agreeing very closely with previous comparisons of SIEMENS' mercury measurements with the B.A. unit.

The determination of the specific resistance of mercury is simple enough in principle, though the execution is somewhat tedious, and the calculation of the results is complicated in practice by the necessity of introducing various temperature corrections. In a first sketch of the method it will be convenient to omit these corrections, which is tantamount to supposing that all the measurements are made at zero. If L be the length and s the section of the column of mercury, R its resistance, r the specific resistance of the metal,

$$\mathrm{R}=\frac{r\mathrm{L}}{s}, \quad \text{or} \quad r=\mathrm{R}\frac{s}{\mathrm{L}}$$

The length L can be measured directly, but s can only be found with the necessary accuracy from the contents. Thus if ρ be the specific gravity of mercury, and W the weight of the whole column in grammes, $\rho\mathrm{L}s=\mathrm{W}$, whence $s=\mathrm{W}/\rho\mathrm{L}$, and

$$r=\frac{\mathrm{RW}}{\rho\mathrm{L}^2}$$

Apart from the temperature corrections already referred to, the simplicity of the formula is disturbed by the inevitable departure from the truly cylindrical form of the glass tubes used to contain the mercury. It is true indeed that to a first order of approximation the formula stands unaltered, as we may see if we understand by s the *mean* section of the tube. The volume is still truly expressed by sL, and the resistance is *approximately* expressed by $r\mathrm{L}/s$. If, however, the squares of the variations of section cannot be neglected, the actual resistance is greater than the formula would lead us to suppose, as is evident if we imagine the section to become at one place very small.

In general we must regard s as a function of the position (x) along the tube at which it is taken. For the purposes of the present paper we may assume with sufficient approximation (see Lord RAYLEIGH's 'Theory of Sound,' § 308)

$$R=r\int\frac{dx}{s}$$

The necessary data with respect to s are obtained by a calibration of the tube. "If a small quantity of mercury is introduced into the tube and occupies a length λ of the tube, the middle point of which is distant x from one end of the tube, then the area s of the section near this point will be $s=C/\lambda$, where C is some constant. The weight of mercury which fills the whole tube is

$$W=\rho\int sdx=\rho C\,\Sigma\left(\frac{1}{\lambda}\right)\frac{L}{n}$$

where n is the number of points at equal distances along the tube, where λ has been measured, and ρ is the mass of unit of volume.

"The resistance of the whole tube is

$$R=\int\frac{rdx}{s}=\frac{r}{C}\Sigma(\lambda)\frac{L}{n}$$

"Hence

$$WR=r\rho\,\Sigma(\lambda)\,\Sigma\left(\frac{1}{\lambda}\right)\frac{L^2}{n^2}$$

and

$$r=\frac{WR}{\rho L^2}\,\frac{n^2}{\Sigma(\lambda)\Sigma\left(\frac{1}{\lambda}\right)}$$

gives the specific resistance of unit of volume" (MAXWELL's 'Electricity,' § 362).

In the sequel

$$\frac{1}{n^2}\Sigma(\lambda)\,\Sigma\left(\frac{1}{\lambda}\right)$$

is denoted by μ; it is a numerical quantity a little greater than unity.

Another correction is required in our method of working to take account of the resistance offered by that part of the mercury in the terminal cups, which is situated just beyond the ends of the tube. The question is identical with that of the correction necessary in calculations of pitch for the open ends of organ pipes (see 'Theory of Sound,' § 307, and Appendix A), and it scarcely admits of absolutely definite solution. We cannot, however, be far wrong in adding to the actual length of the tube ·82 of its diameter, which corresponds to the supposition that the diameter of the mercury column suddenly becomes infinite. Since, in our experiments, the whole correction only amounts to about a thousandth part, even a ten per cent. error in our estimate would scarcely be material.

Let r = resistance of a column of mercury 1 metre long and 1 square millimetre in section, at 0°, expressed in B.A. units.

R = resistance of the tube full of mercury at 0° in B.A. units.

L = length of the tube at t'° in centimetres as measured with brass rod.

l = length of a thread of mercury of nearly the length of the tube at t° as measured with brass rod.

W = weight of the same thread in grammes.

μ = coefficient correcting for conicality of tube.

δL = correction to L on account of the connecting rods not being close up to the ends of the tube $= \cdot 82 \times$ diameter of tube.

ρ = specific gravity of mercury at 0° = 13·595.

γ = cubic expansion of mercury per degree = ·0001795.

g = ,, ,, glass ,, = ·000025.

b = linear expansion of brass ,, = ·000018.

t_0 = temperature of brass measuring rod to which the lengths are corrected = 17°·2.

Then the volume of the thread at 0° = W/ρ

,, ,, $t^{\circ} = \frac{W}{\rho}(1+\gamma t)$

$$\text{Mean section of the tube at } t^{\circ} = \frac{W(1+\gamma t)}{\rho l\{1+b(t-t_0)\}}$$

$$\text{Mean section at } 0^{\circ} = \frac{W(1+\gamma t)}{\rho l\{1+b(t-t_0)\}\{1+\tfrac{2}{3}gt\}}$$

$$\text{Length of the tube at } 0^{\circ} = \frac{(L+\delta L)\{1+b(t'-t_0)\}}{1+\tfrac{1}{3}gt'}$$

$$R = 10^{-4}.r.\mu.\frac{(L+\delta L)\{1+b(t'-t_0)\}}{1+\tfrac{1}{3}gt'}\cdot\frac{\rho l\{1+b(t-t_0)\}\{1+\tfrac{2}{3}gt\}}{W(1+\gamma t)}$$

$$r = \frac{10^4\,RW(1+\gamma t)(1+\tfrac{1}{3}gt')}{\rho\mu\, lL(1+\tfrac{2}{3}gt)}\left(1-\frac{\delta L}{L}\right)\{1-b(t+t'-2t_0)\}$$

The value of ρ is that used by the Committee of the British Association in reducing Dr. MATTHIESSEN's experiments (see reprint of 'Reports on Electrical Standards,' p. 114), and stated to be the mean of the values given by KOPP, REGNAULT, and BALFOUR STEWART. The values of g, γ, and b are taken from EVERETT's 'Units and Physical Constants'—γ being REGNAULT's value for the expansion of mercury. The measurements of the other quantities, which depend on the particular tube used, are given in the following table, together with the resulting value of r. The description of the means employed to obtain these data follows.

Number of observation.	Date of observation, 1882.	Number of tube.	R.	Temperature of coil F.	Temperature of second coil	L in centimetres.	μ.	l in centimetres.	W.	t.	t'.	$\frac{955\times\delta L}{L}$.	$rb(t+t'-2t_0)$	r	Mean values of r from each tube.
1	Feb. 23 & 24	I.	·79912	12°2 11·5	12°1 11·5	87·771	1 00314	87 234	12 442	° 16 5	° 16 5	00103	+·00002	95386	95416
2	„ 25 . .	I.	·79912	12·7	12 5	„	„	87·310	12 4545	17 2	17 2	„	·00000	·95412	
3	„ 21 . .	I.	„	.	.	„	„	87·035	12 4185	18 4	„	„	—·00002	·95424	
4	March 18 .	I.	·79920	13·7	13·75	„	„	87·558	12 486	20·6	„	„	— 00006	95436	
5	Weighed Feb. 14	I.	·79912	.		„	„	87 771	12 523	16 5	16 5	„	+·00002	·95421	
6	Feb. 24 . .	II.	·99088	12 0	.	96·400	1 00007	96 054	12 096	16 7	16 4	·000883	+ 00002	·95389	95419
7	„ 21 to 23	II.	·99081	13·2	.	„	„	95 452	12 0245	16 4	„	„	+·00003	95414	
8	March 7 .	II.	99081	11·5		„	„	95 831	12·074	17·1	„	„	+ 00002	95437	
9	„ 8 .	II	·99079	12·2	.	„	„	96 151	12·113	18 0	18 0	„	—·00003	95436	
10	„ 30 .	II.	·99085	13 25	.								.		
11	„ 6 .	III.	·99711	11·2		123 566	1·00046	122 218	19 620	16 2	18 7	000778	—·00001	95424	95416
12	„ 10 .	III.	·99725	12 9		„	„	123 288	19 780	18 5	„	„	— 00005	95418	
13	„ 13 .	III.	·99720	12 7		„	„	123 221	19 7665	18 4	„	„	—·00005	95399	
14	„ 14 .	III	·99725	13·4		„	„	123 058	19 745	18 3	„	„	—·00005	95425	
15	„ 22 .	IV.	·50783	13 0	12 9	194·137	1·000838	193·410	95·859	14·5	14 5	·000869	+·00009	95440	95427
16	„ 24 .	IV.	·50774	12 7	12 7	„	„	192 576	95 402	16·8		„	+·00005	95415	

Mean of all the above values of r in B A. units ·95418.

The mercury used for all the measurements except 10 and 14 was distilled in vacuo with an apparatus fitted up by Mr. SHAW. In order to see whether a different result might not be obtained with other mercury, some was procured from the chemical laboratory for measurements 10 and 14. For the latter a portion of this mercury was treated with nitric acid and distilled at atmospheric pressure. For measurement 10 it was treated with nitric acid, but not distilled. An accident occurred in carrying out this measurement, so that only the resistance of the column was ascertained; but this agrees so well with the resistances found with the same tube for the other mercury, that there is no reason to suppose that any discrepancy would have appeared in proceeding with the measurement further.

The glass tubes used were supplied by CASSELLA, and were selected for uniformity of bore, so that the correction for conicality should be small. They were slender and easily broken, which made the manipulation of them difficult, and it was in fact owing to a breakage that the tube called No. I. was used so short. The measurements taken with it, at first intended to be preliminary, were, however, made with the same care as in the case of the other tubes, and the difference of length and resistance adds some variety to the data. Tubes II. and III. were cut so that their resistance should be as nearly as possible one B.A. unit. The section of tubes I., II., and III., was approximately 1 square millimetre. Tube IV. was a much larger one, introduced with a view of varying the data as much as could conveniently be done. The diameter of its bore was about 2 millims., and its length was nearly 2 metres. It was cut so as to give a resistance of about half a B.A. unit.

The ends of the tubes were ground into a convex form with emery powder on a lathe, in order that the length (L) of the bore might be measured accurately. This measurement was effected by setting two microscopes, which could be adjusted longitudinally to the exact position required by micrometer-screws graduated to $\frac{1}{10000}$ inch, so that their cross-wires should coincide with the ends of the tube. Observations were made in three or four different positions as the tube was turned round its axis, and the mean taken. After removal of the tube, a brass measuring rod belonging to the British Association was substituted for it, and the number of whole divisions corresponding most nearly to the distance between the cross-wires of the two microscopes was read off. The outstanding fraction of a millimetre was then ascertained by screwing the microscope up to the whole division and reading the difference on the screw-head. For the long tube the measuring rod was too short, and a third microscope had to be used to fix an intermediate point as a fresh departure for the scale. A thermometer laid beside the tube during the measurement gave the temperature (t') at the moment. The brass measuring rod was carefully examined, and its divisions were found to agree among themselves.

The tubes were cleaned by passing through them in succession, by means of a suction-pump, sulphuric acid, nitric acid, caustic potash, and distilled water, followed by air dried with chloride of calcium. The process with omission of the acids was in general repeated between each refilling with mercury, but it was omitted in measurement 7, and there is no record of its having been done in 1, 3, and 6.

To calibrate the tubes a short thread of mercury was inserted and moved to the various positions required, by blowing through a chloride of calcium tube. In the case of tubes I. and II., the length, λ, of the thread was measured by adjusting microscopes to its two ends, with subsequent substitution of an ivory scale divided in fiftieths of an inch. But this method was troublesome; and with tubes III. and IV. the scale was simply placed against the thread and the length read off with a magnifying-glass, a procedure which was found to give sufficiently accurate results, notwithstanding the difficulty arising from parallax owing to the thickness of the glass. The following table gives the different values of λ for each tube.

As a check upon the correction for conicality, two distinct values of μ were in some cases calculated from the alternate observations of λ, and were found to agree closely. It may not be superfluous to mention that in carrying out the computations we must work to six or seven places, although the observed values of λ themselves may not be accurate beyond the third place.

The lengths are in fiftieths of an inch.					
Tube I.		Tube II.	Tube III.		Tube IV.
	80·8	104·5	135·0		171·0
80·0		104·1		134·0	172·0
	77·0	104·5	133·0		171·5
75·8		105·0		132·0	170·5
	76·0	104·5	131·5		171·5
76·4		105·2		130·5	174·5
	75·0	104·3	128·0		175·0
74·0		104·0		127·5	174·5
	73·4	104·7	126·5		175·5
73·0		104·0		126·5	176·5
	72·7	103·0	126·5		177·0
72·3		101·8		126·0	180·0
	72·5		125·0		180·5
71·9				125·5	180·7
	71·1		126·0		182·2
70·1				126·0	183·7
	69·7		126·0		183·5
68·0				126·5	182·5
	67·9		127·0		184·0
67·6				127·0	186·0
	65·9		128·5		186·5
65·3				128·0	
			128·5		
				128·0	

To find the mean section of the tubes we at first tried the method adopted by Messrs. Matthiessen and Hockin in their experiments for the British Association. After aspirating the tube with dry air we placed it in a wooden trough full of mercury, and filled it by suction. It was then held down in the trough with iron weights till it was presumably of the same temperature as the mercury in the trough, which was taken at three places. It was then held by the fingers (previously cooled in other mercury), pressed against its two ends, and taken out of the trough, the mercury adhering to the outside was brushed off, and the contents of the tube were emptied into a small porcelain crucible and weighed. But there was no doubt that when the fingers holding the tube were bare they pressed a little way—how much it was difficult to determine—into the tube, and when they were covered with stiff leather, or other stiff material, it was difficult to get a sufficiently good hold. However, in one case (No. 5) r was calculated from the weight so obtained with leather on the fingers.

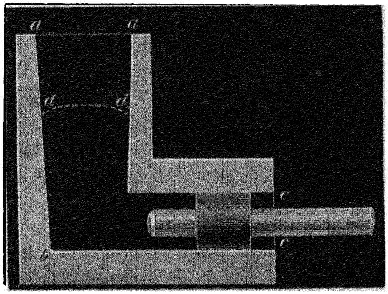
The method, followed by SIEMENS and SABINE, of screwing an iron plate up against the end of the tube, was attempted, but we did not succeed in closing the orifice sufficiently tightly in this way. Ultimately we came to the conclusion that the best results would be obtained by weighing a thread of mercury nearly as long as the tube, and of which we could ascertain the actual length by direct measurement. We thought, also, that there might be some advantage in ascertaining the volume of the mercury from the same filling as that of which the resistance had been taken, as we could not be sure that the closeness of contact between the mercury and the glass was always the same, so that the same volume of mercury would always be contained in the same length of tube, nor that the tube itself was in no way altered by the action of the caustic potash used to clean it. The plan adopted was, therefore, after measuring the resistance, to keep the tube horizontal so as to retain in it most of the mercury while the terminals were removed, and then with microscopes and divided rod to measure the thread of mercury in the same way as the tubes were measured. The length so obtained is called in the table l. The greatest difference between l and L (that in measurement 11) is scarcely over 1 per cent., and in most cases the difference is considerably less, so that, considering how nearly cylindrical the tubes were, the error in the mean section introduced by using a thread of length l instead of L is quite inappreciable. It was another advantage of our method that it avoided the necessity of filling the tube under mercury, which it would have been difficult to do with a tube so long as IV.

The only difficulty in measuring the thread of mercury arose from the convexity of its ends. This was overcome by pressing them flat with little flat-ended vulcanite pins made to fit into the tube. The curvature of the ends when free was not always the same; but it was found that the length of the mercury held with pins varied little from the number calculated on the assumption that the ends were hemispherical, namely, the length of the portion of the column of mercury which was in contact with the glass added to two-thirds of the difference between this length and that between the convex extremities. In some cases, where, owing to the pins not fitting very well or other causes, there was a difficulty in flattening the ends properly, the calculated value was used. A thermometer lay beside the tube during the measurement, so as to give the temperature t. After the measurement, the mercury was blown out into a small crucible and weighed. Care had to be taken not to leave behind minute globules, which, owing probably to the small portion of the tube unoccupied by mercury during the measuring becoming damp from the air of the room or from the fingers, tended to adhere to the glass near the ends.

In three cases (No. 5 as above mentioned and Nos. 3 and 9) the mercury weighed and measured was not that of which the resistance was taken.

No. 3 was done before it occurred to us that there might be an advantage in carrying out both operations with the same filling, and in No. 9 about one-tenth of the mercury was spilt accidentally and had to be replaced.

The equality of the arms of the balance used for the weighing was tested. The weights were compared among themselves and found to be free from appreciable error.



The terminals were composed of L-shaped pieces of ebonite, hollowed out in the manner shown (about full-size) in the figure. Each end of the tube was furnished with a short length of thick rubber tubing, by which the aperture between the glass and the ebonite was closed air-tight. As a further precaution, the space at *c c* beyond the rubber was filled up by pouring in melted paraffine wax.

After the terminals were fitted the tube was again aspirated with dry air through tubes in corks inserted at *a a*, and then filled with mercury, which was poured in to one terminal and allowed to run slowly through to the other till it stood at a considerable height, represented by *d d*, in both terminals. The tube was then placed in a wooden trough and covered with ice. Our reason for using vulcanite terminals rather than glass ones was the fear that under the influence of the ice moisture would collect on the portion of glass above the mercury and serve as a conductor. We certainly avoided all difficulty of this kind by using vulcanite. On the other hand, we probably increased a difficulty which would have existed in any case, namely, that of getting the temperature of the portion of the tube which was within the terminal down to 0°. This portion of the tube was about 2 centims. at each end, or about 5 per cent. of the length in the case of tube I., and about 2 per cent. in the case of tube IV. What the exact temperature of this part of the tube was it is impossible to say, but it was ascertained that the temperature of the mercury in the terminals with the copper connecting rods *in situ* was not higher than 5° or 6°, depending in some degree on the extent to which the ice was piled up round the cup. The mean temperature of the parts of the tubes not directly exposed to ice can hardly have been so high as 2°. Supposing it to have been 2°, and taking the case of tube I., where the largest proportion of the whole length was within the terminals, the effect would be an overestimate of r by about ·00008. In the case of tube IV. the error in r would be less than the half of this.

The tubes were connected with the resistance balance by copper rods, well amalgamated, of which one end stood on the bottom of the vulcanite terminals, so that a considerable portion of the amalgamated copper surface was in contact with the mercury. The rods were kept at a little distance from the ends of the tubes. Dr. MATTHIESSEN brought flattened copper rods up against the ends of his tubes, but this plan appeared open to objection, since it would be very difficult to secure complete contact between the copper and glass all round the edge of the orifice, especially under an opaque fluid like mercury; and any defect in such contact would render necessary an unknown correction. We preferred, therefore, to let the ends of the tube open without obstruction into the mercury cup, which may be regarded as of infinite extent by comparison. The correction necessary to take account of the resistance of the mercury beyond the ends of the tube has already been considered.

The resistance of the rods used to connect I., II., and III. with the bridge was about ·00215 B.A. unit. With tube IV. an additional rod had to be introduced to get the necessary length. This brought the resistance of the rods up to ·00291. The other end of the rods fitted into mercury cups on the resistance balance.

The balance used was one designed by Professor FLEMING (Phil. Mag., ix., p. 109, 1880), in which Professor CAREY FOSTER's method is employed of interchanging the resistances in the two arms of the balance containing the graduated wire, so that the difference between them is expressed in terms of the wire. One thousand divisions of the graduated wire are stated by Professor FLEMING to equal ·0498 B.A. unit, and experiments of our own also showed it to be about ·05. The wire is of platinum-iridium, and as it has a high temperature coefficient compared with the platinum-silver of the standard coils, we thought it undesirable to use much over 100 divisions of it. In order to avoid this in the case of tubes I. and IV. it was necessary to introduce coils from a resistance box in multiple arc. The resistance box employed was one by Messrs. ELLIOTT Brothers. With tube I., 20 ohms from the box were used in multiple arc with the standards against which the tube was balanced, and in the case of tube IV. 24 ohms were used in multiple arc with the tube itself. Tubes II. and III. were balanced against the standard coil belonging to the British Association and deposited at the Cavendish Laboratory, called *F*. For tube IV. another of their unit coils, called the *Flat coil*, was used in multiple arc with *F*. For tube I., *F* and a five-ohm coil were used in multiple arc. The standard coils belonging to the British Association have recently been carefully compared with each other by Professor FLEMING, who has drawn out a chart in which is recorded their variation with temperature, together with their resistance in terms of the mean of their resistances at the temperatures at which they were originally considered to be correct. The values of *F* and of the *Flat coil*—both platinum-silver coils—were taken from this chart. The five-ohm coil had been compared with the British Association standards by ourselves. It was also of platinum-silver, and its temperature coefficient was assumed to be the same as that of the others.

The standard coils were immersed in water whose temperature was observed each time a resistance was measured. These temperatures are given in the table. It may be worth remarking that the resistances were taken in a different room from that in which the lengths were measured, which accounts for the difference between *t* and the temperature of the standards. The thermometer used to find all the temperatures was graduated to fifths, and was corrected by one which had been verified at Kew.

When one coil only was used to balance the tube, its terminals fitted directly into the mercury cups of the bridge, but when two were used in multiple arc their terminals were put into larger mercury cups, which were connected with the mercury cups of the bridge by short copper connecting pieces of about ·00017 ohm resistance.

All the measurements were repeated with reversed battery currents, in order to eliminate thermoelectric disturbance. The readings with battery current each way usually agreed very closely, and the mean of the two was adopted.

It will be observed that the values of R for tube IV. differ by nearly two parts in 10,000, and that there is a less proportional difference, but still an appreciable one, for the other tubes. The greatest actual difference between any two of the values in the table for the same tube is ·00014 ohm. Some small error is due to neglect of the change of resistance of the copper connecting rods and of the bridge wire with temperature. A change of 4° in the temperature of the rods would make a difference of about ·00003 ohm. There is further a probability of error in ascertaining the temperature of the standard coil. A difference of $\frac{1}{10}°$ in this also introduces a difference of ·00003 ohm in the resistance; and there is not only a probable error of perhaps $\frac{1}{10}$ in finding the temperature of the water in which the coil is immersed, but there is no certainty that the coil follows the water exactly. There is evidence, however, that the differences in R are partly due to a real difference in the resistance of different fillings of the tube—whether owing to microscopic bubbles or to a thin varying layer of air between the mercury and the glass, or to what cause, we were unable to determine.*

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* A variation in the closeness of contact between mercury and glass amounting to less than one-fifth of a wave-length of mean light would account for the difference of resistances in the two fillings of tube IV.

We found some reason for thinking that the resistance tended to diminish with time when the mercury remained long in the tube. To examine this we filled tube II. on April 3rd, and found its resistance to be ·99077. It was then left standing full of mercury till April 18th, when the resistance was ·99055. This difference can hardly be relied upon; and in any case the experiments we have tabulated cannot well be affected by any change of this kind, as the interval between the measurement of resistance and that of volume was very short, except in cases 1 and 7. In case 7 the tube stood full of mercury for two days after the resistance was taken. In case 1 the resistance was measured on two successive days, and the mean of the two values taken. The second was the lowest by ·00020, possibly owing to an error. The length was measured immediately after the last measurement of resistance.

The variations in the values of r are, as we should expect, greater than those in R, being affected by probable errors in the other data. The extreme difference amounts to less than 6 in 10,000, and the greatest divergence from the mean value is 3·3 in 10,000.

The mean value of r according to these experiments, ·95418, lies between that deduced from Dr. SIEMENS' experiments for his 1864 standard, namely, ·9534, and Dr. MATTHIESSEN'S value, namely, ·9619 (Phil. Mag., May, 1865), but the difference between our value and Dr. MATTHIESSEN'S, namely, ·00772, is nearly ten times as great as that between ours and Dr. SIEMENS'. We are unable to account satisfactorily for this large difference. One point, however, is worth noting. Dr. MATTHIESSEN measured the resistance of the mercury in his tubes, not at zero, but at temperatures between 18° and 19°·1 (Report of British Association Committee for 1864). To deduce the specific resistance at zero, therefore, he must have assumed the coefficient of variation with temperature, and presumably—though it is nowhere stated in the Report—he used that found from his own experiments (Phil. Trans., 1862), namely, ·074* per cent. per degree. Our own observations have led us to suspect that this value is too small. We made three comparisons of the resistance of tube III. in ice, and in water at approximately the temperature of the room, and one similar comparison with tube IV. The results are given in the following table. Our arrangements were not adapted for observing the resistance at other temperatures, as the open trough afforded no means of checking rapid change.

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* This is the value which results from the experiments made at 0° and at about 20°.

Date.	No of tube.	Mean temperature of water in the trough.	Resistance in water.	Resistance at 0°.	Difference for 1° ÷ resistance at 0°.	Mean of the four values in the last column.
March 13 . . .	III.	12°·7	1·00814	·99720	000863	·000861*
„ 14 .	III.	13 25	1·00874	·99725	000870	
„ 28	III.	12·8	1 00810	·99720	·000854	
„ 24 .	IV	12 5	·51318	·50774	000857	

The above determined mean coincides with the value found by SCHRODER VAN DER KOLK,[†] whose observations, however, related to a much greater range of temperature. An observation by WERNER SIEMENS[‡] between the temperature 18°·5 and 0° gives for the coefficient ·00090.

The difference between the coefficients ·00074 and ·00086, as applied to the reduction from 18°·7 (the mean temperature of the tubes in Dr. MATTHIESSEN'S observations) to 0°, would account for about one quarter of the difference between his results and our own.

The remainder of the discrepancy may possibly be. connected with the manner in which Dr. MATTHIESSEN'S tubes were calibrated. Although in the description of the process a *small* column of mercury is spoken of (Reprint, p. 128), it is distinctly stated on the preceding page that the lengths of the columns of mercury were 383, 291, 245 millims. respectively, *i.e.*, nearly half the lengths of the tubes. It is possible that this may be a mistake; but if such lengths were really used, the correction for conicality would have been much underestimated, so that the specific resistance of mercury would come out too high. In the case of uniform conicality the true correction would be four times as great as that obtained by applying the formula applicable to short threads, to cases where the length is about half that of the tube.

[*January*, 1883.—The measuring rod and the weights used in the above investigation have been compared with standards verified by the Board of Trade, and the errors have been found to be negligible. But since the value of ρ employed relates to weighings *in vacuo*, a corresponding correction is called for here. On this account the final number, ·95418, should be reduced to

·95412.]

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* It should be noticed that the resistances here compared are those of the contents of a certain glass tube at various temperatures, so that the accompanying temperature variations of length and section are determined by the properties of glass and not by the properties of mercury. The results are therefore not quite comparable with those obtained in similar experiments with solid metallic wires, which are free to determine for themselves their length and section.

[†] POGG. Ann , cx., 1860.

[‡] Ibid., cxiii., 1861.

III. *On the Ultra-Violet Spectra of the Elements.*—Part I. *Iron* (*with a map*).

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[Plates 1–5.]

Ångstrom's "normal solar spectrum" has served most spectroscopists as a standard of reference for wave-lengths in the visible part of the spectrum. Cornu's continuation of it, and particularly the map of the iron lines which he used in the construction of it, serves very well for such a standard up to the limit of the solar spectrum, *i.e.*, to a wave-length 2948 (U). For the region above this we have had to use Mascart's and Cornu's wave-lengths of the cadmium lines, and Cornu's wave-lengths of some magnesium lines. The intervals between those lines are, however, too great for any close approximation to the wave-lengths of intervening lines by interpolation, and, moreover, the wave-lengths did not appear to be determined with sufficient precision to serve as a standard, and the lines are ill adapted to that purpose by reason of their diffuse character. For the determination of the wave-lengths of lines in this higher region we have, therefore, been driven to form a standard for ourselves. For this purpose we have chosen the iron spectrum which had been employed by Cornu in the region which he mapped, and seemed to answer the purpose well, both from the number of lines which it presents and their characteristic grouping by which they may easily be recognised by anyone who has once become familiar with them. The wave-lengths of the most prominent lines were determined by means of a Rutherfurd diffraction grating, as detailed below, between the wave-lengths 2948 and 2327; but beyond this there is a remarkable falling off in the intensity of the iron lines, and between wave-lengths 2327 and 2135 (which is near the limit of transparency of Iceland spar) we have preferred to determine the wave-lengths of the prominent copper lines which are numerous and strong in that region. The wave-lengths of a series of lines at short intervals having thus been determined, those of the intervening lines were obtained by interpolation, and the result is shown in the map of the iron spectrum above U which accompanies this paper.

The instruments.

The goniometer used was constructed for us by HILGER, and has a circle of 18 inches diameter, graduated at intervals of 5′ by SIMMS. Fractions of 5′ are read by a microscope with a micrometer eye-piece fixed to the arm which carries the telescope.

The telescope and collimator have each an object-glass consisting of a single lens of quartz 1½ inch diameter and a focal length of 18½ inches for the sodium yellow light, but not more than 16 inches for the highest rays measured. The sliding tubes of both telescope and collimator are graduated in fiftieths of an inch, and alterations of focus were made on both instruments at the same time, so that the rays falling on the grating might always be nearly parallel. The graduation of the sliding tube was also used for ascertaining the distance of the photographic plate from the object-glass of the telescope. This was necessary for computing the corrections of the angular measure, as explained below. The collimator is furnished with a quartz lens, of 3 inches focal length, in front of the slit, movable to a greater or less distance, but retained by guides so that its axis may remain coincident with that of the collimator. This lens was placed about 6 inches in front of the slit, and the source of light at the same distance beyond it, so that its image was focussed on the slit.

The measurements were all made by means of photographs taken on WRATTEN and WAINWRIGHT's instantaneous dry gelatine plates. The plates (2¼ inches by 1 inch) were held in a small slide attached to a tube which fitted the telescope in place of the eye-piece, and thus the plate could easily be turned about an axis perpendicular to its plane and coinciding with the axis of the telescope. This turning of the plate about is a matter of no small importance, as it enabled us to avoid the errors which would have arisen from measuring the distances of the lines from the irregular edge of the plate, as will be seen when the mode of measuring the photographs is described. The plates were retained in one position in the slide during exposure by three springs, of which two pressed against two edges of the plate and the other against its back.

The grating was ruled on speculum metal by CHAPMAN with RUTHERFURD's machine, and has a ruled surface of rather more than 1¾ inch in each dimension, with 17,296 lines to the inch. It is an excellent grating, but, of course, has the faults which belong to the particular machine by which it was ruled. The definition, when it has not been exposed to variations of temperature, is very good, but it has one inconvenience for our present purpose, which is, that the focus for the same ray in the spectrum of the same order does not fall at quite the same distance from the object-glass of the telescope on the two sides of the normal. The explanation of this has been given by CORNU (Comptes Rendus, lxxx., 645), who has shown that it is due to a systematic variation in the distance between the ruled lines. As the method employed by us depends upon taking angular measures of the position of the ray on both sides of the normal, and any shift of the focussing tube between the two positions would be likely to introduce serious errors, we have been obliged to be content with the photograph taken on one side being a little, though only a little, less sharp in definition than that taken on the other side. The grating was used with its plane perpendicular to the axis of the collimator, and it was brought into that position in the following way. The telescope and collimator having their axes directed as nearly as possible on to the centre of the circle, the telescope was placed opposite the collimator and the cross wires brought on to the image of the slit, and the reading of the circle taken. The grating was then placed in position and adjusted for level until the spectra occupied the middle of the field of view in all positions of the telescope. It was then adjusted in azimuth until the images of the D lines, from the light of a sodium flame in front of the slit, in the spectra of the second order, on the two sides of the normal were at equal angular distances from the axis of the collimator, as determined by the reading taken when the telescope was opposite the collimator. The grating was then clamped in that position. This adjustment had to be made by hand and was liable to disturbance in the clamping, so that it was afterwards found that the plane of the grating was not quite perpendicular to the axis of the collimator; but as the errors arising from this in the measures on the two sides nearly compensate one another, the final error in the wave-length from this cause is very small indeed.

For measuring the photographs a micrometer was constructed for us by HILGER. This is attached to the stage of a microscope and carries a small frame in which the photographic plate is held by springs. The micrometer-screw has 100 turns to the inch, and by the drum-head $\frac{1}{100}$th of a turn or $\frac{1}{10000}$th of an inch can be read. A 1 inch object-glass to the microscope was used, and measures were made by moving the plate until the lines of the photograph were successively bisected by a spider line in the eye-piece. The reading of the micrometer gave the distances between the lines.

The source of light employed was, in the first instance, the arc from a DE MERITENS magneto-electric machine, in a crucible of magnesia into which iron wire was introduced. But from the overlapping of the spectra of different orders, and the large amount of light emitted by the arc, we found that the plates were so clouded in many places that the lines could not be well seen, and we abandoned the arc for the spark between iron electrodes. This was produced by a large induction coil, worked by 5 GROVE'S cells, and having a large Leyden jar connected with the secondary wire. No inconvenience arose from the overlapping of the different orders when the spark was used, because the parts of the spectra of higher and lower orders which overlapped the part of the spectrum of the fourth order to be measured were always considerably out of focus, the object glasses of telescope and collimator being uncorrected, and so the light of the lines in the overlapping spectra was diffused and produced only a faint clouding of the plate, which in no way interfered with the measure of the lines of the fourth order.

Mode of proceeding.

The first thing to be done was to obtain a focussing scale for the different angles, including the portion of the spectrum to be mapped. This was already known approximately from SARASIN's table of refractive indices of quartz, and was corrected by a series of trial plates taken at successive angles.

The electrodes, between which the spark was passed, were arranged so that the spark should pass horizontally (the slit being vertical), and at such a height that the visible image formed by the lens in front of the slit on the plates of the slit might fall just above, or partly above, the centre of the slit. The lower half of the slit was closed by a shutter, so that only the lower half of the field of view of the telescope was illuminated. The distance of the spark from the lens had next to be adjusted, as the focal length of the lens for the visible rays is very different from that for the ultra-violet which were to be photographed. This was done by estimation, as there was no need to have the image exactly focussed on the slit, so long as the slit was in the middle of the image and light enough passed through.

The telescope was then set to such an angle as would bring the line so measured nearly into the centre of the field, the focus adjusted, the photographic slide adjusted and levelled, and the plate exposed. An image of the lines was thus formed on the lower half of the plate. The slide was then turned round through 180° about the axis of the tube, so as to bring what had before been the upper side of the plate to the lower side and right to left, and again levelled. The plate was then again exposed and thus a second image of the line impressed, and one of the two images was as far to the right of the axis about which the plate had been turned as the other was to the left. Half the distance between the two images would therefore be the distance of the line from the centre of the field, and the knowledge of this would give the means of calculating the deviation of the rays producing the line from the axis of the telescope. The telescope was next turned to the corresponding angle on the other side of the collimator and the operations repeated with a second plate, but without any alterations of adjustment. The telescope was then moved through a small angle, generally 5′ or 10′, and the same operations as before repeated on both sides of the collimator, the second pair of plates being intended to serve as a check upon the first.

Similar operations were then repeated at such angular intervals as should bring in the most characteristic strong lines of iron all along the scale. Beyond the wave-length 2327, it was found that the iron lines were too faint to produce any sufficient impression on the plates. For the region beyond this up to wave-length 2135, copper electrodes were substituted for iron. This being about the limit of transparency of calcite (the material of our prisms), was the limit of our study of spectra at this time.

The measurement of the distance between the two images of a line was made by the micrometer above described, and to convert this distance into arc, it was necessary to know the distance of the plate from the centre of the object-glass of the telescope. It was found by measurement that the distance from the object-glass to the front of the photographic plate, when the sliding tube was at the 100th division, was 439 millims., and the thickness of the lens at its middle 2·5 millims., which, divided by the refractive index of quartz for the high rays observed is very nearly 1·5 millim., making the distance of the photographic plate from the optical centre of the object-glass very nearly 440·5 millims. From this the formula $\tan^{-1}\frac{D}{3068{\cdot}5+4a}$ was deduced for the angular distance of a line from the axis of the telescope when the difference of the micrometer readings of the two images of the line is D, and the number of divisions of the scale of sliding tube at which the telescope was focussed was a. To determine whether the angle so found was to be added or subtracted from the reading of the circle, all that was necessary was to observe whether the micrometer reading of the upper or lower image as seen in the microscope were the greater. The angular position of the line having thus been found on each side of the collimator, the wave-length was deduced by the ordinary formula.

The following tables give the measurements made, the calculated angles, and the wave-lengths deduced.

The quantities recorded in the several columns is as follows :—

I. The mark of the particular photographic plate.

II. The reading of the circle, giving the position of the telescope.

III. The reading of the scale of focussing tube.

IV. The measured distance of the two images of the line in hundredths of an inch.

V. One half this distance reduced to angular measure with the sign + or − according as it is to be added or subtracted from the reading of the circle to give the angular position of the line.

VI. The angular position of the line.

VII. The mean values of the angular positions so found. When several sets of plates have been taken at different times, or with varied adjustments, the means for the several sets are given separately.

VIII. The deviation from the normal to the grating of the line as deduced.

IX. The wave-length deduced or adopted. When different sets of plates give different values, these different values are inserted in brackets.

The figures belonging to different lines are separated by horizontal spaces.

In the case of plates marked with a figure less than 172 the arc was employed to give the light, in the case of all plates with the mark 172 and upwards the spark was employed.

I.	II	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
341	28 30 0	49	10 63	−11 12	28 18 48			
342	309 30 0	49	8 43	+ 8 53	309 38 53			
343	309 25 0	49	13 39	+14 6	309 39 6	309 39 0	39 19 54	2326·9
341	28 30 0	49	6 11	− 6 26	28 23 34			
342	309 30 0	49	3 96	+ 4 10	309 34 10			
343	309 25 0	49	8·72	+ 9 11	309 34 11		39 24 42	2330·9
341	28 30 0	49	4 36	− 4 35	28 25 25			
342	309 30 0	49	2 01	+ 2 7	309 32 7			
343	309 25 0	49	6 92	+ 7 17	309 32 17	309 32 12	39 26 36	2332·5
341	28 30 0	49	1 64	+ 1 44	28 31 44			
342	309 30 0	49	3 96	− 4 10	309 25 50			
343	309 25 0	49	0 87	+ 0 55	309 25 55	309 25 52	39 32 56	2337 7
341	28 30 0	49	7·92	+ 8 20	28 38 20			
342	309 30 0	49	10 37	−10 55	309 19 5			
343	309 25 0	49	5 57	− 5 52	309 19 8	309 19 6	39 39 37	2343·2
341	28 30 0	49	8·78	+ 9 15	28 39 15			
342	309 30 0	49	11 27	−11 52	309 18 8			
343	309 25 0	49	6 36	− 6 42	309 18 18	309 18 13	39 40 31	2343·9
341	28 30 0	49	13·40	+14 7	28 44 7			
342	309 30 0	49	15 89	−16 44	309 13 16			
343	309 25 0	49	11 07	−11 39	309 13 21	309 13 14	39 45 27	2348 0
262	28 45 0	50	11 67	+12 16	28 57 16			
263	28 40 0	50	16·38	+17 14	28 57 14	28 57 15		
264	309 15 0	50	13 97	−14 42	309 0 18			
265	309 10 0	50	9 18	− 9 39	309 0 21	309 0 19	39 58 28	2358 7
262	28 45 0	50	12 91	+13 35	28 58 35			
263	28 40 0	50	17 63	+18 33	28 58 33	28 58 34		
264	309 15 0	50	15 15	−15 56	308 59 4			
265	309 10 0	50	10·44	−10 59	308 59 1	308 59 2	39 59 46	2359·7
262	28 45 0	50	18·33	+19 17	29 4 17			
263	28 40 0	50	23 02	+24 13	29 4 13	29 4 15		
264	309 15 0	50	20 68	−21 45	308 53 15			
265	309 10 0	50	15·98	−16 48	308 53 12	308 53 13	40 5 31	2364·4
81	29 30 3	25	1·70	+ 1 51	29 31 54			
82	29 30 3	25	2 02	+ 2 12	29 32 15			
83	29 34 3	25	2 37	− 2 35	29 31 28	29 31 52		
84	308 59 40	25	1·52	− 1 39	308 58 1			
85	308 59 40	25	1·20	− 1 18	308 58 22	308 58 12	40 16 50	(2373·7)
100	308 53 50	25	2·47	− 2 41	308 51 9			
101	308 53 50	25	2·66	− 2 53	308 50 57	308 51 8		
102	29 40 0	25	14·64	−15 53	29 24 7			
103	29 30 0	25	5·40	− 5 52	29 24 8	29 24 7	40 16 32	2373·4
262	28 45 0	50	28·65	+30 8	29 15 8			
263	28 40 0	50	33·33	+35 3	29 15 3	29 15 6		
264	309 15 0	50	31·09	−32 42	308 42 18			
265	309 10 0	50	26·36	−27 43	308 42 17	308 42 17	40 16 25	(2373·3)

I.	II	III	IV	V.	VI	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
262	28 45 0	50	38·19	+40 10	29 25 10			
263	28 40 0	50	42 96	+45 11	29 25 11	29 25 11		
264	309 15 0	50	40 87	−42 59	308 32 1			
265	309 10 0	50	36 11	−37 59	308 32 1		40 26 35	(2381·6)
81	29 30 3	25	11 29	+12 14	29 42 17			
82	29 30 3	25	11·02	+12 36	29 42 39			
86	29 34 3	25	7·13	+ 7 44	29 41 47	29 42 14		
84	308 59 40	25	11 02	−11 57	308 47 43			
85	308 59 40	25	10 82	−11 44	308 47 56	308 47 48	40 27 13	(2382·1)
100	308 53 50	25	12 00	−13 1	308 40 49			
101	308 53 50	25	12 26	−13 18	308 40 32	308 40 40		2381·7
102	29 40 0	25	5·27	− 5 43	29 34 17			
103	29 30 0	25	4 10	+ 4 27	29 34 27	29 34 22	40 26 51	(2381·8)
258	308 15 0	53	16 34	+17 7	308 32 7			
259	308 20 0	53	11·86	+12 26	308 32 26	308 32 17		
260	29 45 0	53	18 78	−19 41	29 25 19			
261	29 40 0	53	13 70	−14 21	29 25 39	29 25 29	40 26 36	(2381·6)
81	29 30 3	25	18 96	+20 34	29 50 37			
82	29 30 3	25	19·14	+20 55	29 50 58			
86	29 34 3	25	14 66	+15 54	29 49 57	29 50 31		
84	308 59 40	25	18·66	−20 15	308 39 25			
85	308 59 40	25	18 45	−20 1	308 39 39	308 39 32	40 35 29	(2388·8)
258	308 15 0	53	8 56	+ 8 58	308 23 58			
259	308 20 0	53	4 01	+ 4 12	308 24 12	308 24 5		
260	29 45 0	53	11·02	−11 33	29 33 27			
261	29 40 0	53	5·89	− 6 10	29 33 50	29 33 38	40 34 46	(2388·2)
100	308 53 50	25	19 61	−21 17	308 32 33			
101	308 53 50	25	19 77	−21 27	308 32 23	308 32 28		
102	29 40 0	25	2 31	+ 2 30	29 42 30			
103	29 30 0	25	11 64	+12 38	29 42 38	29 42 34	40 35 8	2388·5
100	308 53 50	25	27·60	−29 57	308 23 53			
101	308 53 50	25	27·79	−30 9	308 23 4	308 23 47		
102	29 40 0	25	10 33	+11 12	29 51 12			
103	29 30 0	25	19 64	+21 18	29 51 18	29 51 15	40 43 44	(2395·5)
258	308 15 0	53	0·22	+ 0 14	308 15 14			
259	308 20 0	53	4·41	− 4 37	308 15 23	308 15 18		2395·4
260	29 45 0	53	2 85	− 3 3	29 41 57			
261	29 40 0	53	2 20	+ 2 18	29 42 18	29 42 7	40 43 24	(2395·2)
100	308 53 50	25	31·80	−34 30	308 19 20			
101	308 53 50	25	32·03	−34 45	308 19 5	308 19 12		
102	29 40 0	25	14·46	+15 41	29 55 41			
103	29 30 0	25	23·85	+25 53	29 55 53	29 55 47	40 48 18	(2399 2)
258	308 15 0	53	4·16	− 4 22	308 10 38			
259	308 20 0	53	8·65	− 9 4	308 10 56	308 10 47		2399
260	29 45 0	53	1·45	+ 1 31	29 46 31			
261	29 40 0	53	6·53	+ 6 51	29 46 51	29 46 41	40 47 57	(2398·9)
258	308 15 0	53	10·89	−11 25	308 3 35			
259	308 20 0	53	15·48	−16 13	308 3 47	308 3 41		
260	29 45 0	53	8·08	+ 8 28	29 53 28			
261	29 40 0	53	13·18	+13 46	29 53 46	29 53 37	40 54 58	2404·5

I.	II.	III	IV	V.	VI.	VII	VIII.	IX
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
258	308 15 0	53	12 97	−13 36	308 1 24			
259	308 20 0	53	17 53	−18 22	308 1 38	308 1 31		
260	29 45 0	53	10 18	+10 40	29 55 40			
261	29 40 0	53	15 18	+15 54	29 55 54	29 55 47	40 57 8	2406 3
258	308 15 0	53	17 57	−18 25	307 56 35			
259	308 20 0	53	22 12	−23 11	307 56 49	307 56 42		
260	29 45 0	53	14 65	+15 21	30 0 21			
261	29 40 0	53	19 69	+20 38	30 0 38	30 0 30	41 1 54	2410·2
258	308 15 0	53	18 20	−19 4	307 55 56			
259	308 20 0	53	22 79	−23 53	307 56 7	307 56 1		
260	29 45 0	53	15 41	+16 9	30 1 9			
261	29 40 0	53	20 35	+21 20	30 1 20	30 1 15	41 2 37	2410·7
258	308 15 0	53	20 93	−21 56	307 53 4			
259	308 20 0	53	25 55	−26 46	307 53 14	307 53 9		
260	29 45 0	53	17 99	+18 51	30 3 51			
261	29 40 0	53	23 01	+24 7	30 4 7	30 3 59	41 5 25	2413·0
254	30 45 0	57	7·82	− 8 9	30 36 51			
255	30 40 0	57	3 05	− 3 11	30 36 49	30 36 50		
256	307 15 0	57	5·71	+ 5 57	307 20 57			
257	307 20 0	57	1 06	+ 1 6	307 21 6	307 21 2	41 37 54	2439·0
254	30 45 0	57	1 60	− 1 40	30 43 20			
255	30 40 0	57	3·17	+ 3 18	30 43 18	30 43 19		
256	307 15 0	57	0·67	− 0 42	307 14 18			
257	307 20 0	57	5 37	− 5 36	307 14 24	307 14 21	41 44 29	2444·3
254	30 45 0	57	0 27	− 0 17	30 44 43			
255	30 40 0	57	4·45	+ 4 38	30 44 38	30 44 41		
256	307 15 0	57	1 98	− 2 4	307 12 56			
257	307 20 0	57	6 70	− 6 59	307 13 1	307 12 58	41 45 52	2445·4
250	32 0 0	65	47 16	−48 42	31 11 18			
251	306 5 0	65	40·20	+41 31	306 46 31		42 12 23	2466·4
250	32 0 0	65	42·39	−43 47	31 16 13			
251	306 5 0	65	35 41	+36 34	306 41 34		42 17 19	2470·3
250	32 0 0	65	30·79	−31 48	31 28 12			
253	31 45 0	65	16·06	−16 35	31 28 25	31 28 18		
251	306 5 0	65	23 53	+24 18	306 29 18			
252	306 15 0	65	13 80	+14 15	306 29 15	306 29 17	42 29 31	2480 0
250	32 0 0	65	28 43	−29 22	31 30 38			
253	31 45 0	65	13 78	−14 14	31 30 46	31 30 42		
251	306 5 0	65	21·27	+21 58	306 26 58			
252	306 15 0	65	11 42	+11 48	306 26 48	306 26 53	42 31 54	2481·8
250	32 0 0	65	27·73	−28 38	31 31 22			
251	306 5 0	65	20 52	+21 12	306 26 12		42 32 35	2482·4
71	31 54 45	50	3·07	− 3 14	31 51 31			
72	306 38 55	50	5·78	+ 6 5	306 45 0			
73	306 38 55	50	5·79	+ 6 5	306 45 0	306 45 0	42 33 15	2482 9

I.	II.	III.	IV.	V.	VI.	VII	VIII	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length
71	31 54 45	50	2 04	− 2 9	31 52 36			
72	306 38 55	50	4·77	+ 5 1	306 43 56			
73	306 38 55	50	4·85	+ 5 6	306 44 1	306 43 59	42 34 18	2483·7
250	32 0 0	65	23·27	−24 . 2	31 35 58			
253	31 45 0	65	8 35	− 8 37	31 36 23	31 36 10		
251	306 5 0	65	16 24	+16 46	306 21 46			
252	306 15 0	65	6·02	+ 6 13	306 21 13	306 21 30	42 37 20	2486·1
107	306 10 10	50	17·40	+18 18	306 28 28			
108	306 0 0	50	27 46	+28 53	306 28 53			
109	306 10 10	50	17 71	+18 36	306 28 46			
110	306 20 0	50	8 64	+ 9 6	306 29 6	306 28 48		
111	32 0 5	50	11 83	−12 27	31 47 33			
112	32 0 5	50	12 02	−12 39	31 47 26	31 47 33	42 39 23	2487·7
113	31 50 10	50	2 39	− 2 31	31 47 39			
71	31 54 45	50	4·74	+ 4 59	31 59 44			
72	306 38 55	50	2 06	− 2 10	306 36 45			
73	306 38 55	50	2 08	− 2 11	306 36 44	306 36 44	42 41 30	(2489·4)
172	31 50 5	50	2 00	− 2 6	31 47 59			
173	31 40 20	50	7·41	+ 7 48	31 48 8			
174	32 0 5	50	11 67	−12 16	31 47 49	31 47 59		
175	306 23 0	50	1 04	+ 1 6	306 24 6			2489·5
176	306 33 5	50	8 37	− 8 48	306 24 17			
177	306 13 0	50	10 74	+11 18	306 24 18	306 24 14	42 41 53	(2489 7)
250	32 0 0	65	19 12	−19 45	31 40 15			
253	31 45 0	65	4 34	− 4 29	31 40 31	31 40 23		
251	306 5 0	65	11 64	+12 1	306 17 1			
252	306 15 0	65	1·79	+ 1 54	306 16 54	306 16 57	42 41 43	2489 5
250	32 0 0	65	17 91	−18 30	31 41 30			
253	31 45 0	65	3 12	− 3 13	31 41 47	31 41 38		
251	306 5 0	65	10·51	+10 51	306 15 51			
252	306 15 0	65	0·73	+ 0 45	306 15 45	306 15 48	42 42 55	2490·5
250	32 0 0	65	14 87	−15 21	31 44 39			
253	31 45 0	65	0·09	− 0 6	31 44 54	31 44 46		
251	306 5 0	65	7·42	+ 7 40	306 12 40			
252	306 15 0	65	2·41	− 2 29	306 12 29	306 12 35	42 46 5	2493·0
172	31 50 5	50	2·19	+ 2 18	31 52 23			
173	31 40 20	50	11·59	+12 11	31 52 31			
174	32 0 5	50	7·55	− 7 56	31 52 9	31 52 21		
175	306 23 0	50	3 19	− 3 22	306 19 38			2493·0
176	306 33 5	50	12·59	−13 15	306 19 50			
177	306 13 0	50	6·53	+ 6 52	306 19 52	306 19 47	42 46 17	(2493·1)
246*	305 20 0	65	51·22	+52 54	306 12 54			
247	305 25 0	65	46·37	+47 53	306 12 53	306 12 54		
248	32 45 0	65	58·23	−60 8	31 44 52			
249	32 40 0	65	53 28	−55 1	31 44 59	31 44 55	42 46 0	(2492·9)

I.	II.	III.	IV	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
246*	305 20 0	65	44 18	+45 38	306 5 38			
247	305 25 0	65	39·34	+40 38	306 5 38			
248	32 45 0	65	51 14	−52 49	31 52 11			
249	32 40 0	65	46 31	−47 50	31 52 11		42 53 16	(2498·6)
172	31 50 5	50	8·97	+ 9 26	31 59 31			
173	31 40 20	50	18·45	+19 24	31 59 44			
174	32 0 5	50	0·64	− 0 40	31 59 25	31 59 33		
175	306 23 0	50	10·13	−10 39	306 12 21			2498·7
176	306 33 5	50	19 56	−20 34	306 12 31			
177	306 13 0	50	0 42	− 0 27	306 12 33	306 12 30	42 53 32	(2498 8)
91	306 39 0	50	26 86	−28 15	306 10 45			
92	31 55 0	50	24 04	+25 17	32 20 17			
93	31 55 0	50	23 92	+25 10	32 20 10	32 20 14	43 4 45	2507 6
181	305 22 0	55	33·07	+34 34	305 56 34	305 56 34		
182	32 56 10	55	38·61	−40 22	32 15 48			
183	32 55 5	55	37 47	−39 10	32 15 55			
184	32 51 15	55	33 88	−35 25	32 15 50			
185	32 35 2	55	18 40	−19 14	32 15 48			
186	32 25 15	55	9·02	− 9 26	32 15 49	32 15 50	43 9 38	2511·4
246*	305 20 0	65	28 14	+29 4	305 49 4			
247	305 25 0	65	23·20	+23 54	305 48 54	305 48 59		
248	32 45 0	65	35 35	−36 31	32 8 29			
249	32 40 0	65	30·54	−31 33	32 8 27	32 8 28	43 9 44	2511·4
246*	305 20 0	65	10 95	+11 19	305 31 19			
247	305 25 0	65	5·97	+ 6 10	305 31 10	305 31 15		
248	32 45 0	65	18 43	−19 2	32 25 58			
249	32 40 0	65	13·56	−14 0	32 26 0	32 25 59	43 27 22	(2525·2)
181	305 22 0	55	16 10	+16 50	305 38 50	305 38 50		
182	32 56 10	55	22 02	−23 1	32 33 9			
183	32 55 5	55	20 82	−21 46	32 33 19			
184	32 51 15	55	17 24	−18 1	32 33 14			
185	32 35 2	55	1·72	− 1 48	32 33 14			
186	32 25 15	55	7 69	+ 8 2	32 33 17	32 33 15	43 27 13	2525·1
246*	305 20 0	65	9 81	+10 8	305 30 8			
247	305 25 0	65	4 84	+ 5 0	305 30 0	305 30 4		
248	32 45 0	65	17·35	−17 55	32 27 5			
249	32 40 0	65	12·48	−12 53	32 27 7	32 27 6	43 28 31	2526·1
246*	305 20 0	65	5 72	+ 5 54	305 25 54			
247	305 25 0	65	0 78	+ 0 48	305 25 48	305 25 51		
248	32 45 0	65	13·25	−13 41	32 31 19			
249	32 40 0	65	8·32	− 8 36	32 31 24	32 31 22	43 32 45	(2529·8)
178	305 53 10	55	19 06	−19 55	305 33 15			
179	305 47 55	55	13 90	−14 32	305 33 23			
180	305 18 0	55	14 84	+15 31	305 33 31			
181	305 22 0	55	10 92	+11 25	305 33 25	305 33 23		
182	32 56 10	55	16·92	−17 42	32 38 28			
183	32 55 5	55	15 70	−16 25	32 38 40			
184	32 51 15	55	12 23	−12 47	32 38 28			
185	32 35 2	55	3 42	+ 3 35	32 38 37			
186	32 25 15	55	12·87	+13 27	32 38 42	32 38 35	43 32 36	2529·2

I.	II.	III.	IV	V	VI	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
246*	305 20 0	65	0 55	+ 0 34	305 20 34			
247	305 25 0	65	4 39	− 4 32	305 20 28	305 20 31		
248	32 45 0	65	8 22	− 8 29	32 36 31			
249	32 40 0	65	3 27	− 3 23	32 36 37	32 36 34	43 38 2	2533·4
246*	305 20 0	65	0·35	− 0 22	305 19 38			
247	305 25 0	65	5 31	− 5 29	305 19 29	305 19 31		
248	32 45 0	65	7 20	− 7 26	32 37 34			
249	32 40 0	65	2 32	− 2 24	32 37 36	32 37 35	43 39 0	2534·2
246*	305 20 0	65	3 41	− 3 31	305 16 29			
247	305 25 0	65	3·27	− 3 33	305 16 27	305 16 28		
248	32 45 0	65	4 24	− 4 23	32 40 37			
249	32 40 0	65	0 65	+ 0 40	32 40 40	32 40 38	43 42 5	2536 6
178	305 53 10	55	28 22	−29 30	305 23 40			
179	305 47 55	55	23·04	−24 5	305 23 50			
180	305 18 0	55	5 71	+ 5 58	305 23 58			
181	305 22 0	55	1·74	+ 1 52	305 23 52	305 23 50		
182	32 56 10	55	7 90	− 8 15	32 47 55			
183	32 55 5	55	6·67	− 6 58	32 48 7			
184	32 51 15	55	3·20	− 3 21	32 47 54			
185	32 35 2	55	12 36	+12 55	32 47 57			
186	32 25 15	55	21 79	+22 47	32 48 2	32 47 59	43 42 5	2536 6
178	305 53 10	55	30·92	−32 19	305 20 51			
179	305 47 55	55	25·71	−26 53	305 21 2			
180	305 18 0	55	3 11	+ 3 15	305 21 15			
181	305 22 0	55	0 91	− 0 57	305 21 3	305 21 3		
182	32 56 10	55	5·40	− 5 39	32 50 31			
183	32 55 5	55	4 13	− 4 19	32 50 46			
184	32 51 15	55	0·76	− 0 48	32 50 27			
185	32 35 2	55	14 95	+15 38	32 50 40			
186	32 25 15	55	24 30	+25 24	32 50 39	32 50 37	43 44 47	2538 6
187	33 10 10	62	18·64	−19 19	32 50 51			
188	33 20 15	62	28 24	−29 16	32 50 59	32 50 55		
189	305 3 0	62	17 86	+18 31	305 21 31			
190	304 52 40	62	27 59	+28 36	305 21 16	305 21 23	43 44 46	2538 6
243	33 25 0	65	40,47	−41 48	32 43 12			
244	33 15 0	65	30 78	−31 47	32 43 13			
248	32 45 0	65	1 61	− 1 40	32 43 20			
249	32 40 0	65	3 27	+ 3 23	32 43 23	32 43 17		
245	304 50 0	65	23·15	+23 55	305 13 55			
246	304 45 0	65	27 85	+28 46	305 13 46			
246*	305 20 0	65	6 01	− 6 12	305 13 48			
247	305 25 0	65	11 01	−11 22	305 13 38	305 13 47	43 44 45	2538 6
242	33 30 0	65	32·13	−33 11	32 56 49			
243	33 25 0	65	27·25	−28 9	32 56 51			
244	33 15 0	65	17 63	−18 13	32 56 47	32 56 49		
245	304 50 0	65	9 83	+10 9	305 0 9			
246	304 45 0	65	14 56	+15 2	305 0 2	305 0 6	43 58 21	2549·1
242	33 30 0	65	31·41	−32 26	32 57 34			
243	33 25 0	65	26·58	−27 27	32 57 33			
244	33 15 0	65	16·87	−17 26	32 57 34	32 57 34		
245	304 50 0	65	8 99	+ 9 17	304 59 17			
246	304 45 0	65	13·77	+14 13	304 59 13	304 59 15	43 59 10	2549·7

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
242	33 30 0	65	30·64	−31 39	32 58 21			
243	33 25 0	65	25·75	−26 36	32 58 24			
244	33 15 0	65	16·12	−16 39	32 58 21	32 58 22		
245	304 50 0	65	8·19	+ 8 28	304 58 28			
246	304 45 0	65	12·89	+13 19	304 58 19	304 58 23	43 59 59	2550·3
242	33 30 0	65	15·64	−16 9	33 13 51			
243	33 25 0	65	10·74	−11 6	33 13 54			
244	33 15 0	65	1·13	− 1 10	33 13 50	33 13 52		
245	304 50 0	65	6·97	− 7 12	304 42 48			
246	304 45 0	65	2·27	− 2 21	304 42 39	304 42 43	44 15 34	2562·3
188	33 20 15	62	1·31	+ 1 21	33 21 36	33 21 36		
189	305 3 0	62	11·86	−12 18	304 50 42			
190	304 52 40	62	2·25	− 2 20	304 50 20	304 50 31	44 15 33	2562·3
242	33 30 0	65	14·50	−14 59	33 15 1			
243	33 25 0	65	9·57	− 9 53	33 15 7			
244	33 15 0	65	0·10	+ 0 6	33 15 6	33 15 5		
245	304 50 0	65	7·96	− 8 13	304 41 47			
246	304 45 0	65	3·45	− 3 34	304 41 26	304 41 36	44 16 44	2563·2
193	303 19 55	57	57·80	+60 16	304 20 11			
194	34 53 15	57	58·93	−61 27	33 51 48		44 45 48	(2585·3)
195	34 43 5	54	49·10	−51 23	33 51 42			
196	303 30 0	54	47·72	+49 57	304 9 57		44 45 52	2585·4
242	33 20 0	65	13·75	+14 12	33 44 12			
243	33 25 0	65	18·57	+19 11	33 44 11			
244	33 15 0	65	28·30	+29 14	33 44 14	33 44 12		
245	304 50 0	65	36·59	−37 47	304 12 13			
246	304 45 0	65	32·00	−33 3	304 11 57	304 12 5	44 46 4	(2585·5)
96	303 19 57	65	45·14	+46 37	304 6 34			
97	303 19 57	65	44·91	+46 26	304 6 23	304 6 29		
98	34 50 0	65	38·13	−39 23	34 10 37			
99	34 50 0	65	37·99	−39 14	34 10 46	34 10 41	45 2 6	2597·6
96	303 19 57	65	43·80	+45 14	304 5 11			
97	303 19 57	65	43·63	+45 4	304 5 1	304 5 6		
98	34 50 0	65	36·71	−37 55	34 12 5			
99	34 50 0	65	36·49	−37 41	34 12 19	34 12 12	45 3 33	(2598·7)
195	34 43 5	54	32·03	−33 32	34 9 33			
196	303 30 0	54	30·34	+31 45	304 1 45		45 3 54	(2599·0)
193	303 19 55	57	40·38	+42 2	304 1 58			2598·8
194	34 53 15	57	41·75	−43 32	34 9 43		45 3 52	(2599·0)
195	34 43 5	54	22·35	−23 24	34 19 41			
196	303 30 0	54	20·59	+21 33	303 51 33		45 14 4	(2606·6)
193	303 19 55	57	30·45	+31 45	303 51 40			2606·7
194	34 53 15	57	31·77	−33 8	34 20 7		45 14 13	(2606·8)
193	303 19 55	57	24·28	+25 19	303 45 14			
194	34 53 15	57	25·84	−26 57	34 26 18		45 20 32	(2611·5)

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length
195	34 43 5	54	16·24	−17 0	34 26 5			
196	303 30 0	54	14·57	+15 15	303 45 15		45 20 25	2611·4
96	303 19 57	65	27·72	+28 38	303 48 35			
97	303 19 57	65	27·54	+28 27	303 48 24	303 48 30		
98	34 50 0	65	20·89	−21 34	34 28 26			
99	34 50 0	65	20 75	−21 26	34 28 34	34 28 30	45 20 0	(2611·1)
96	303 19 57	65	25 14	+25 58	303 45 55			
97	303 19 57	65	24·93	+25 45	303 45 42	303 45 49		
98	34 50 0	65	18 34	−18 57	34 31 3			
99	34 50 0	65	18 33	−18 56	34 31 4	34 31 3	45 22 37	2613·1
96	303 19 57	65	20·18	+20 50	303 40 47			
97	303 19 57	65	20 05	+20 42	303 40 39	303 40 43		
98	34 50 0	65	13 41	−13 51	34 36 9			
99	34 50 0	65	13·28	−13 43	34 36 17	34 36 13	45 27 45	(2616·9)
195	34 43 5	54	8 94	− 9 21	34 33 44			
196	303 30 0	54	7 18	+ 7 31	303 37 31		45 28 6	2617·2
193	303 19 55	57	16·92	+17 39	303 37 33			
194	34 53 15	57	18 54	−19 20	34 33 55		45 28 11	2617·2
193	303 19 55	57	6 58	+ 6 52	303 26 47			
194	34 53 15	57	8·29	− 8 39	34 44 36		45 38 55	(2625 3)
195	34 43 5	54	1·18	+ 1 14	34 44 19			
196	303 30 0	54	3 14	− 3 17	303 26 43		45 38 48	2625 2
96	303 19 57	65	9·76	+10 5	303 30 2			
97	303 19 57	65	9 45	+ 9 46	303 29 43	303 29 53		
98	34 50 0	65	2·96	− 3 4	34 46 56			
99	34 50 0	65	2 84	− 2 55	34 47 5	34 47 0	45 38 33	(2625 0)
237	34 30 0	73	7·12	+ 7 17	34 37 17			
238	34 25 0	73	11 88	+12 9	34 37 9	34 37 11		
239	303 0 0	70	18 88	+19 19	303 19 19	303 19 18	45 38 56	(2625·3)
240	303 15 0	70	4·18	+ 4 17	303 19 17			
241	34 45 0	70	7·68	− 7 53	34 37 7			
96	303 19 57	65	6·18	+ 6 23	303 26 20			
97	303 19 57	65	5·91	+ 6 6	303 26 3	303 26 12		
98	34 50 0	65	0·52	+ 0 32	34 50 32			
99	34 50 0	65	0·63	+ 0 39	34 50 39	34 50 35	45 42 12	(2627·7)
193	303 19 55	57	3·12	+ 3 20	303 23 15			
194	34 53 15	57	4·94	− 5 9	34 48 6		45 42 25	2627 9
237	34 30 0	73	10·63	+10 53	34 40 53			
238	34 25 0	73	15 41	+15 46	34 40 46	34 40 47		
239	303 0 0	70	15·44	+15 48	303 15 48	303 15 45	45 42 31	2627 9
240	303 15 0	70	0·66	+ 0 41	303 15 41			
241	34 45 0	70	4·16	− 4 16	34 40 44			

I.	II.	III.	IV	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
195	34 43 5	54	4·62	+ 4 51	34 47 56			
196	303 30 0	54	6·53	− 6 50	303 23 10		45 42 23	2627·9
96	303 19 57	65	2·49	+ 2 34	303 22 31			
97	303 19 57	65	2·23	+ 2 18	303 22 15	303 22 23		
98	34 50 0	65	4·34	+ 4 29	34 54 29			
99	34 50 0	65	4·45	+ 4 36	34 54 36	34 54 33	45 46 5	(2630·6)
193	303 19 55	57	0·61	− 0 38	303 19 17			
194	34 53 15	57	1·41	− 1 28	34 51 47		45 46 15	2630·7
195	34 43 5	54	8·15	+ 8 32	34 51 37			
196	303 30 0	54	10·07	−10 32	303 19 28		45 46 5	(2630·6)
237	34 30 0	73	14·27	+14 36	34 44 36			
238	34 25 0	73	18·99	+19 26	34 44 26	34 44 29		
239	303 0 0	70	11·58	+11 51	303 11 51	303 11 55	45 46 17	(2630·8)
240	303 15 0	70	2·93	− 3 1	303 11 59			
241	34 45 0	70	0·59	− 0 36	34 44 24			
195	34 43 5	54	51·32	+53 36	35 36 41			
196	303 30 0	54	53·71	−56 13	302 33 47		46 31 27	(2664·2)
232	302 30 0	76	2·77	− 2 49	302 27 11			
234	302 35 0	76	7·66	− 7 48	302 27 12	302 27 11		
235	35 30 0	76	0·16	− 0 10	35 29 50		46 31 20	2664·1
323	302 25 0	76	2·84	+ 2 54	302 27 54			
324	35 30 0	76	0·00	0 0	35 30 0		46 31 3	(2663·9)
195	34 43 5	54	53·96	+56 28	35 39 33			
196	303 30 0	54	56·38	−59 0	302 31 0		46 34 17	(2666·3)
323	302 25 0	76	0·23	+ 0 14	302 25 14			
324	35 30 0	76	2·57	+ 2 37	35 32 37		46 33 42	(2665·8)
232	302 30 0	76	5·41	− 5 31	302 24 29			
234	302 35 0	76	10·32	−10 31	302 24 29	302 24 29		
235	35 30 0	76	2·51	+ 2 34	35 32 34		46 34 3	2666·1
225	37 0 0	79	61·41	−62 22	35 57 38			
228	36 30 0	79	32·14	−32 39	35 57 21	35 57 29		
226	301 5 0	79	53·53	+54 22	301 59 22			
227	301 35 0	79	24·15	+24 32	301 59 32	301 59 27	46 59 1	(2684·4)
232	302 30 0	76	29·89	−30 28	301 59 32			
234	302 35 0	76	34·72	−35 23	301 59 37	301 59 34		2684·2
235	35 30 0	76	26·71	+27 14	35 57 14			
236	35 35 0	76	21·87	+22 18	35 57 18	35 57 16	46 58 51	(2684·3)
323	302 25 0	76	24·21	−24 41	302 0 19			
324	35 30 0	76	26·82	+27 21	35 57 21		46 58 31	(2684·0)

I.	II.	III.	IV.	V.	VI.	VII	VIII.	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
232	302 30 0	76	40 61	−41 24	301 48 36			
234	302 35 0	76	45·34	−46 12	301 48 48	301 48 42		
235	35 30 0	76	37 31	+38 2	36 8 2			
236	35 35 0	76	32 46	+33 5	36 8 5	36 8 4	47 9 41	2692·1
323	302 25 0	76	34·83	−35 30	301 49 30			
324	35 30 0	76	37 46	+38 11	36 8 11		47 9 21	(2691·9)
225	37 0 0	79	50·81	−51 36	36 8 24			
228	36 30 0	79	21·49	−21 50	36 8 10	36 8 17		2692·1
226	301 5 0	79	42 77	+43 26	301 48 26			
227	301 35 0	79	13 40	+13 37	301 48 37	301 48 32	47 9 52	(2692·2)
225	37 0 0	79	35·43	−35 59	36 24 1			
228	36 30 0	79	6 09	− 6 11	36 23 49	36 23 55		
226	301 5 0	79	27 19	+27 37	301 32 37			
227	301 35 0	79	2 23	− 2 16	301 32 44	301 32 41	47 25 37	2703·6
197	300 59 55	65	25 63	+26 28	301 26 23			
198	301 10 0	65	15·74	+16 16	301 26 16	301 26 20		
200	37 3 0	65	16 83	−17 23	36 45 37			
201	37 12 55	65	26 64	−27 31	36 45 24	36 45 30	47 39 35	(2713 7)
225	37 0 0	79	21·26	−21 36	36 38 24			
228	36 30 0	79	8·10	+ 8 14	36 38 14	36 38 19		2713·8
226	301 5 0	79	12·95	+13 9	301 18 9			
227	301 35 0	79	16·48	−16 21	301 18 39	301 18 24	47 39 58	(2714 0)
225	37 0 0	79	3 43	− 3 29	36 56 31			
228	36 30 0	79	26 04	+26 27	36 56 27	36 56 29		
226	301 5 0	79	5 05	− 5 8	300 59 52			
227	301 35 0	79	34 46	−35 0	301 0 0	300 59 56	47 58 16	(2727 1)
197	300 59 55	65	7·78	+ 8 2	301 7 57			
198	301 10 0	65	2 14	− 2 13	301 7 57	301 7 57		2727 0
200	37 3 0	65	0·88	+ 0 55	37 3 55			
201	37 12 55	65	8·75	− 9 2	37 3 53	37 3 54	47 57 58	(2726 9)
197	300 59 55	65	8·68	− 8 58	300 50 57			
198	301 10 0	65	18·58	−19 11	300 50 49	300 50 55		
200	37 3 0	65	17 27	+17 50	37 20 50			
201	37 12 55	65	7 59	+ 7 50	37 20 45	37 20 48	48 14 57	(2739 0)
229	37 30 0	86	16·07	−16 11	37 13 49			2739·1
230	300 35 0	86	7·97	+ 8 2	300 43 2			
231	300 30 0	86	12·87	+12 58	300 42 58	300 43 0	48 15 24	(2739 3)
229	37 30 0	86	10·99	−11 4	37 18 56			
230	300 35 0	86	2 88	+ 2 54	300 37 54			
231	300 30 0	86	7 81	+ 7 52	300 37 52	300 37 58	48 20 31	(2743 0)
197	300 59 55	65	13 79	−14 15	300 45 40			
198	301 10 0	65	23 62	−24 24	300 45 36	300 45 38		2742·8
200	37 3 0	65	22·23	+22 57	37 25 57			
201	37 12 55	65	12 47	+12 53	37 25 48	37 25 53	48 20 8	(2742·7)

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
197	300 59 55	65	18·35	−18 57	300 40 58			
198	301 10 0	65	28·21	−29 18	300 40 52	300 40 55		(2746·0)
200	37 3 0	65	26·68	+27 33	37 30 33			
201	37 12 55	65	17·04	+17 36	37 30 31	37 30 32	48 24 49	2746·1
229	37 30 0	86	6·46	− 6 31	37 23 29			
230	300 35 0	86	1·71	− 1 43	300 33 17			
231	300 30 0	86	3·27	+ 3 18	300 33 18	300 33 17	48 25 6	(2746·2)
229	37 30 0	86	5·78	− 5 49	37 24 11			
230	300 35 0	86	2·38	− 2 24	300 32 36			
231	300 30 0	86	2·56	+ 2 35	300 32 35	300 32 36	48 25 47	2746·7
197	300 59 55	65	18·99	−19 37	300 40 18			
198	301 10 0	65	28·85	−29 48	300 40 12	300 40 15		
200	37 3 0	65	27·38	+28 17	37 31 17			
201	37 12 55	65	17·69	+18 39	37 31 34	37 31 26	48 25 36	(2746·6)
197	300 59 55	65	22·23	−22 58	300 36 57			
198	301 10 0	65	32·12	−33 10	300 36 50	300 36 53		
200	37 3 0	65	30·57	+31 34	37 34 34			
201	37 12 55	65	20·97	+22 7	37 35 2	37 34 48	48 28 58	(2749·1)
229	37 30 0	86	2·51	− 2 32	37 27 28			
230	300 35 0	86	5·65	− 5 41	300 29 19			
231	300 30 0	86	0·74	− 0 45	300 29 15	300 29 17	48 29 6	2749·0
128	301 4 55	67	27·80	−28 30	300 36 16			
129	301 4 55	67	27·61	−28 27	300 36 28			
131	300 54 55	67	18·32	−18 53	300 36 2	300 36 15		
125	37 20 5	67	15·66	+16 18	37 36 13			
124	37 30 5	70	6·09	+ 6 15	37 36 20	37 36 17	48 30 1	2749·7
229	37 30 0	86	2·96	+ 2 59	37 32 59			
230	300 35 0	86	11·25	−11 20	300 23 40			
231	300 30 0	86	6·27	− 6 19	300 23 41	300 23 40	48 34 39	2753·0
229	37 30 0	86	6·42	+ 6 28	37 36 28			
230	300 35 0	86	14·71	−14 49	300 20 11			
231	300 30 0	86	9·74	− 9 49	300 20 11	300 20 11	48 38 9	2755·5
197	300 59 55	65	31·13	−32 9	300 27 44			
198	301 10 0	65	41·00	−42 21	300 27 39	300 27 41		
200	37 3 0	65	39·44	+40 44	37 43 44			
201	37 12 55	65	29·76	+31 23	37 44 18	37 44 1	48 38 10	2755·5
197	300 59 55	65	47·45	−49 0	300 10 55			
198	301 10 0	65	57·22	−59 6	300 10 54	300 10 54		
200	37 3 0	65	55·57	+57 23	38 0 23			
201	37 12 55	65	45·88	+48 23	38 1 18	38 0 51	48 54 58	(2767·3)
220	299 35 0	79	27·90	+28 20	300 3 20			
221	299 35 0	79	27·82	+28 15	300 3 15	300 3 18		
222	38 30 0	79	36·45	−37 1	37 52 59			
218	38 30 0	79	36·32	−36 53	37 53 7			
219	38 25 0	79	31·56	−32 3	37 52 57	37 53 2	48 54 52	2767·2
286	38 20 0	84	26·85	−27 7	37 52 53			
287	299 40 0	84	23·15	+23 23	300 3 23		48 54 45	(2767·1)

I.	II.	III.	IV.	V.	VI	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length
286	38 20 0	84	10·20	−10 18	38 9 42			
287	299 40 0	84	6 27	+ 6 20	299 46 20		49 11 41	2779·0
220	299 35 0	79	11·26	+11 27	299 46 27			
221	299 35 0	79	11·16	+11 20	299 46 20	299 46 24		
222	38 30 0	79	19 96	−20 16	38 9 44			
218	38 30 0	79	19·78	−20 5	38 9 55			
219	38 25 0	79	14·98	−15 13	38 9 47	38 9 51	49 11 43	2779·0
220	299 35 0	79	5·06	+ 5 8	299 40 8			
221	299 35 0	79	4 95	+ 5 2	299 40 2	299 40 5		
222	38 30 0	79	13 80	−14 1	38 15 59			
218	38 30 0	79	13 62	−13 51	38 16 9			
219	38 25 0	79	8 84	− 8 59	38 16 1	38 16 3	49 17 59	2783·4
286	38 20 0	84	3 97	− 4 1	38 15 59			
287	299 40 0	84	0·05	+ 0 3	299 40 3		49 17 58	2783 4
321	39 0 0	86	20·58	−20 43	38 39 17			
322	298 55 0	86	23·65	+23 49	299 18 49		49 40 14	2798 8
321	39 0 0	86	0 52	− 0 31	38 59 29			
322	298 55 0	86	3·26	+ 3 17	298 58 17		50 0 36	2812 9
321	39 0 0	86	25·93	+26 7	39 26 7			
322	298 55 0	86	23 26	−23 26	298 31 34		50 27 17	2831·1
281	298 25 0	87	5·95	+ 5 59	298 30 59			
284	298 25 0	87	5·81	+ 5 51	298 30 51	298 30 55		
282	39 35 0	87	9·27	− 9 20	39 25 40			
283	39 35 0	87	9 74	− 9 48	39 25 12			
285	39 35 0	87	9 60	− 9 40	39 25 20	39 25 24	50 27 14	(2831 0)
281	298 25 0	87	0·13	− 0 3	298 24 52			
284	298 25 0	87	0·36	− 0 22	298 24 38	298 24 45		
282	39 35 0	87	3 20	− 3 13	39 31 47			
283	39 35 0	87	2 71	− 2 44	39 32 16			
285	39 35 0	87	3·58	− 3 36	39 31 24	39 31 49	50 33 32	2835·3
281	298 25 0	87	7·47	− 7 31	298 17 29			
284	298 25 0	87	7 59	− 7 38	298 17 22	298 17 25		
282	39 35 0	87	3 99	+ 4 1	39 39 1			
283	39 35 0	87	4 54	+ 4 34	39 39 34			
285	39 35 0	87	3 58	+ 3 36	39 38 36	39 39 5	50 40 50	2840 3
288	297 25 0	88	25 62	+25 45	297 50 45			
289	40 35 0	88	30 28	−30 26	40 4 26		51 6 51	(2857·8)
318	297 20 0	88	31·35	+31 30	297 51 30			2857·9
319	297 20 0	88	31 40	+31 33	297 51 33	297 51 32		
320	40 30 0	88	23 73	−23 51	40 6 9		51 7 18	(2858 1)
278	40 35 0	88	29 57	−29 43	40 5 17			
279	297 25 0	88	26·13	+26 16	297 51 16			
280	297 25 0	88	26 34	+26 28	297 51 28	297 51 22	51 6 57	(2857·8)

I.	II.	III.	IV.	V	VI.	VII.	VIII	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
278	40 35 0	88	8 62	− 8 40	40 26 20			
280	297 25 0	88	5·29	+ 5 19	297 30 19		51 28 0	(2871·9)
288	297 25 0	88	4 54	+ 4 34	297 29 34			
289	40 35 0	88	9 45	− 9 30	40 25 30		51 27 58	(2871 9)
318	297 20 0	88	10 37	+10 25	297 30 25			2872·0
319	297 20 0	88	10 34	+10 24	297 30 24	297 30 25		
320	40 30 0	88	2 77	− 2 47	40 27 13		51 28 24	(2872 2)
269	40 50 0	91	21 93	−21 58	40 28 2			
270	40 55 0	91	27·00	−27 2	40 27 58	40 28 0		
271	297 10 0	91	18 96	+18 59	297 28 59			
272	297 5 0	91	23 94	+23 59	297 28 59		51 29 30	(2872·9)
288	297 25 0	88	3·02	+ 3 2	297 28 2			
289	40 35 0	88	7·86	− 7 54	40 27 6		51 29 32	(2872·9)
317	297 25 0	88	3 94	+ 3 58	297 28 58			2873·0
318	297 20 0	88	8 93	+ 8 59	297 28 59			
319	297 20 0	88	8 86	+ 8 54	297 28 54	297 28 57		
320	40 30 0	88	1·36	− 1 22	40 28 38		51 29 51	(2873·1)
278	40 35 0	88	7 12	− 7 9	40 27 51			
279	297 25 0	88	3·47	+ 3 29	297 28 29			
280	297 25 0	88	3·78	+ 3 48	297 28 48	297 28 39	51 29 36	2873·0
278	40 35 0	88	3 82	+ 3 50	40 38 50			
280	297 25 0	88	7 58	− 7 37	297 17 23		51 40 48	2880·4
269	40 50 0	91	10 94	−10 58	40 39 2			
270	40 55 0	91	15 99	−16 1	40 38 59	40 39 0		
271	297 10 0	91	7 66	+ 7 40	297 17 40			
272	297 5 0	91	12 56	+12 35	297 17 35	297 17 38	51 40 41	(2880·3)
288	297 25 0	88	8 12	− 8 10	297 16 50			
289	40 35 0	88	3 18	+ 3 12	40 38 12		51 40 41	(2880 3)
317	297 25 0	88	7 28	− 7 19	297 17 41			
318	297 20 0	88	2 56	− 2 34	297 17 26			
319	297 20 0	88	2 46	− 2 28	297 17 32	297 17 33		
320	40 30 0	88	9·75	+ 9 48	40 39 48		51 41 7	(2880 6)
269	40 50 0	91	6 50	− 6 31	40 43 29			
270	40 55 0	91	11 55	−11 34	40 43 26	40 43 27		
271	297 10 0	91	3 21	+ 3 13	297 13 18			
273	297 5 0	91	8 14	+ 8 12	297 13 12	297 13 12	51 45 7	2883 3
278	40 35 0	88	8 17	+ 8 13	40 43 13			
279	297 25 0	88	11 92	−11 59	297 13 1			
280	297 25 0	88	11 84	−11 54	297 13 6	297 13 3	51 45 5	2883 3
288	297 25 0	88	12·53	−12 36	297 12 24			
289	40 35 0	88	7·52	+ 7 34	40 42 34		51 45 5	2883·3

I.	II.	III.	IV.	V.	VI.	VII.	VIII	IX
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
317	297 25 0	88	11 59	−11 39	297 13 21			
318	297 20 0	88	6 95	− 6 59	297 13 1			
319	297 20 0	88	6·77	− 6 48	297 13 12	297 13 11		
320	40 30 0	88	14 13	+14 12	40 44 12	.	51 45 31	(2883·5)
275	296 10 0	93	2 48	− 2 29	296 7 32			
276	41 50 0	93	0 91	− 0 55	41 49 5			
277	41 55 0	93	6·00	− 6 1	41 48 59	41 49 2	52 50 45	(2926·1)
267	296 10 0	93	2 85	− 2 51	296 7 9			
268	41 50 0	93	1·20	− 1 12	41 48 48	..	52 50 50	2926 0
310	41 55 0	93	10 19	+10 11	42 5 11			
311	41 50 0	93	15 00	+14 59	42 4 59	42 5 5		
312	296 0 0	93	8 16	− 8 9	295 51 51	..	53 6 37	2936·3
267	296 10 0	93	30 58	−30 33	295 39 27			
268	41 50 0	93	26 50	+26 29	42 16 29	..	53 18 31	(2943·9)
275	296 10 0	93	30·48	−30 27	295 39 33			
276	41 50 0	93	26 78	+26 45	42 16 45			
277	41 55 0	93	21 73	+21 43	42 16 43	42 16 44	53 18 36	2944 0
310	41 55 0	93	21·95	+21 56	42 16 56			
311	41 50 0	93	26 75	+26 44	42 16 44	42 16 50		
312	296 0 0	93	20 04	−20 1	295 39 59			
315	296 0 0	93	19 97	−19 57	295 40 3	295 40 1	53 18 25	(2943·9)
275	296 10 0	93	35 50	−35 28	295 34 32			
276	41 50 0	93	31 94	+31 55	42 21 55			U
277	41 55 0	93	26·89	+26 52	42 21 52	42 21 54	53 23 41	2947·3

As in the highest region we have used the lines in the spectrum of the spark between copper electrodes as lines of reference, we add here the measurements and calculated values for these lines, and for the strong magnesium lines which we have sometimes used in interpolating.

COPPER LINES.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
327	312 55 0	24	27 88	+30 17	313 25 17			
330	313 5 0	24	13·19	+19 46	313 24 46			
331	313 5 0	24	18 81	+20 26	313 25 26	313 25 10		
328	25 0 0	24	23 96	−26 2	24 33 58			
329	24 45 0	24	10·30	−11 11	24 33 49	24 33 53	35 34 21	2135 7
327	312 55 0	24	13·87	+15 4	313 10 4			
330	313 5 0	24	4·17	+ 4 32	313 9 32			
331	313 5 0	24	4 73	+ 5 8	313 10 8	313 9 55		
328	25 0 0	24	10·15	−11 2	24 48 58			
329	24 45 0	24	3·53	+ 3 50	24 48 50	24 48 54	35 49 30	2148·9

COPPER LINES—(continued).

I.	II.	III	IV	V.	VI	VII.	VIII.	IX.
	° ′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
297	25 50 0	31	24 22	−26 5	25 23 55			
298	312 10 0	31	23 66	+25 29	312 35 29		36 24 13	2178·8
297	25 50 0	31	13 18	−14 11	25 35 49			
298	312 10 0	31	12 24	+13 11	312 23 11		36 36 19	2189·2
325	26 0 0	31	22 94	−24 42	25 35 18			
326	311 55 0	31	25·83	+27 49	312 22 49		36 36 14	2189 2
325	26 0 0	31	20 02	−21 33	25 38 27			
326	311 55 0	31	22 98	+24 45	312 19 45		36 39 21	2191·8
297	25 50 0	31	10 32	−11 7	25 38 53			
298	312 10 0	31	9 37	+10 5	312 20 5		36 39 24	(2191·9)
297	25 50 0	31	2 37	− 2 33	25 47 27			
298	312 10 0	31	1·46	+ 1 34	312 11 34		36 47 56	2199 2
325	26 0 0	31	0 65	− 0 42	25 59 18			
326	311 55 0	31	3·19	+ 3 26	311 58 26		37 0 26	(2209 9)
291	26 25 0	37	23 62	−25 15	25 59 45			
293	26 35 0	37	33 52	−35 55	25 59 5			
296	26 25 0	34	23 27	−24 55	26 0 5			
297	25 50 0	31	9·08	+ 9 47	25 59 47	25 59 40		
298	312 10 0	31	10 19	−10 58	311 59 2			
299	311 10 0	39	46 93	+50 2	312 0 2			2209·7
294	311 25 0	37	31 70	+33 53	311 58 53			
295	311 35 0	34	22 73	+24 23	311 59 23	311 59 20	37 0 10	(2209·6)
291	26 25 0	37	15·26	−16 19	26 8 41			
293	26 35 0	37	24 69	−26 27	26 8 33			
296	26 25 0	34	14·80	−15 53	26 9 7			
297	25 50 0	31	17·55	+18 54	26 8 54			
300	26 50 0	39	39·24	−41 50	26 8 10	26 8 41		
294	311 25 0	37	22 94	+24 31	311 49 31			
295	311 35 0	34	14 04	+15 4	311 50 4			2217·5
298	312 10 0	31	18 77	−20 13	311 49 47			
299	311 10 0	39	37 91	+40 25	311 50 25	311 49 57	37 9 22	
325	26 0 0	31	7 68	+ 8 16	26 8 16			
326	311 55 0	31	5 40	− 5 49	311 49 11		37 9 33	(2217·6)
291	26 25 0	37	3 49	− 3 44	26 21 16			
293	26 35 0	37	12·99	−13 55	26 21 5			
296	26 25 0	34	3 05	− 3 16	26 21 44			
300	26 50 0	39	27·23	−29 2	26 20 58	26 21 16		
294	311 25 0	37	10 95	+11 42	311 36 42			
295	311 35 0	34	1 87	+ 2 0	311 37 0			
299	311 10 0	39	25·85	+27 33	311 37 33	311 37 5	37 22 5	2228 3
293	26 35 0	37	11 58	−12 24	26 22 36			
296	26 25 0	34	1·65	− 1 46	26 23 14	26 22 55		
294	311 25 0	37	9 51	+10 10	311 35 10			
295	311 35 0	34	0·51	+ 0 32	311 35 32			
299	311 10 0	39	24·52	+26 8	311 36 8	311 35 37	37 23 39	2229·6

Copper Lines—(continued).

I.	II.	III.	IV	V	VI	VII.	VIII	IX.
	′ ″		hundredths of an inch.	′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
291	26 25 0	37	11 59	+12 23	26 37 23			
292	26 30 0	38	7 30	+ 7 47	26 37 47			
293	26 35 0	37	2 15	+ 2 18	26 37 18			
296	26 25 0	34	12 00	+12 52	26 37 52			
300	26 50 0	39	11 52	−12 17	26 37 43	26 37 37		
294	311 25 0	37	4·54	− 4 51	311 20 9			
295	311 35 0	34	13 28	−14 15	311 20 45			
299	311 10 0	39	10·15	+10 49	311 20 49	311 20 34	37 38 31	2242·2
332	27 0 0	39	20 54	−21 54	26 38 6			
333	26 55 0	39	15 96	−17 1	26 37 59	26 38 2		
334	310 55 0	39	24 32	+25 56	311 20 56			
335	311 0 0	39	19·70	+20 49	311 20 49	311 20 53	37 38 34	2242 2
332	27 0 0	39	15·77	−16 49	26 43 11			
333	26 55 0	39	11 03	−11 46	26 43 14	26 43 12		
334	310 55 0	39	19·47	+20 46	311 15 46			
335	311 0 0	39	14 61	+15 27	311 15 27	311 15 37	37 43 47	2246·6
291	26 25 0	37	16 48	+17 37	26 42 37			
292	26 30 0	38	12 16	+12 59	26 42 59			
293	26 35 0	37	7·03	+ 7 32	26 42 32			
296	26 25 0	34	16 80	+18 1	26 43 1			
300	26 50 0	39	6·62	− 7 3	26 42 57	26 42 49		
294	311 25 0	37	9·45	−10 6	311 14 54			
295	311 35 0	34	18 20	−19 31	311 15 29			
299	311 10 0	39	5 31	+ 5 40	311 15 40	311 15 21	37 43 44	2246·6
332	27 0 0	39	2 98	+ 3 10	27 3 10			
333	26 55 0	39	7 80	+ 8 19	27 3 19	27 3 15		
335	311 0 0	39	4 27	− 4 31	310 55 29		38 3 53	2263·6
332	27 0 0	39	16 90	+18 1	27 18 1			
333	26 55 0	39	21 60	+23 2	27 18 2			
334	310 55 0	39	13·52	−14 25	310 40 35			
335	311 0 0	39	18 18	−19 13	310 40 47			
336	310 30 0	46	10 33	+10 55	310 40 55			
337	310 25 0	46	15 12	+15 59	310 40 59	310 40 49		
338	27 30 0	46	11 24	−11 53	27 18 7			
340	27 35 0	46	16 00	−16 55	27 18 5			
339	27 25 0	46	6 71	− 7 6	27 17 54	27 18 2	38 18 37	2276·0
291	26 25 0	37	48·59	+51 56	27 16 56			
292	26 30 0	38	44 59	+47 36	27 17 36			
293	26 35 0	37	39·99	+42 51	27 17 51			
296	26 25 0	34	48 64	+52 11	27 17 11			
300	26 50 0	39	26·44	+28 11	27 18 11	27 17 38		
294	311 25 0	37	42 01	−44 54	310 40 6			
295	311 35 0	34	50 70	−54 23	310 40 37			
299	311 10 0	39	27·31	−29 7	310 40 53	310 40 32	38 18 33	(2275·9)
336	310 30 0	46	10 33	−10 55	310 19 5			
337	310 25 0	46	5 55	− 5 52	310 19 8	310 19 6		
338	27 30 0	46	9 30	+ 9 50	27 39 50			
340	27 35 0	46	4 40	+ 4 39	27 39 39			
339	27 25 0	46	13·76	+14 33	27 39 33	27 39 41	38 40 19	2294·1

MAGNESIUM LINES.

I	II	III	IV.	V.	VI	VII.	VIII.	IX
	° ′ ″		hundredths of an inch.	° ′ ″	° ′ ″	° ′ ″	° ′ ″	wave-length.
129	301 4 55	67	68 67	−1 10 45	299 54 10			
128	301 4 55	67	68 85	−1 10 58	299 53 57			
131	300 54 55	67	59 32	−1 1 7	299 53 48	299 54 4		
134	300 4 55	75	10 67	−0 10 53	299 54 2			
135	300 4 55	75	10 89	−0 10 36	299 54 19			
139	300 0 3	72	5 66	−0 5 47	299 54 16			
140	300 0 3	72	5·95	−0 6 6	299 53 57			
137	38 10 0	75	8 75	+0 8 56	38 18 56			
138	38 15 0	72	3 42	+0 8 30	38 18 30			
125	37 20 5	67	56 82	+0 58 32	38 18 37			
124	37 30 5	70	47 33	+0 48 35	38 18 40	38 18 41	49 12 19	2779 4
137	38 10 0	75	30 88	+0 31 31	38 41 31			
138	38 15 0	72	25 68	+0 26 18	38 41 18	38 41 25		
139	300 0 3	72	27 86	−0 28 32	299 31 31			
140	300 0 3	72	28 02	−0 28 42	299 31 21	299 31 26	49 35 0	2795 2
306	299 15 0	86	8 75	+0 8 49	299 23 49	299 23 50		
307	38 45 0	86	10 89	−0 11 1	38 33 59			
308	38 30 0	85	3 84	+0 3 52	38 33 52	38 33 55		
309	299 30 0	85	6 10	−0 6 9	299 23 51		49 35 2	2795 2
137	38 10 0	75	41 09	+0 41 56	38 51 56			
138	38 15 0	72	35 83	+0 36 42	38 51 42	38 51 49		
139	300 0 3	72	37 82	−0 38 44	299 21 19			
140	300 0 3	72	38·21	−0 39 8	299 20 55	299 21 7	49 45 21	2802 4
306	299 15 0	86	1 57	−0 1 35	299 13 25			
307	38 45 0	86	0 59	−0 0 36	38 44 24			
308	38 30 0	85	14·16	+0 14 17	38 44 17	38 44 21		
309	299 30 0	85	16 48	−0 16 37	299 13 23	299 13 24	49 45 28	(2802·4)
303	40 0 0	85	2·97	−0 3 0	39 57 0			
304	298 0 0	85	0 34	+0 0 21	298 0 21			
305	298 5 0	85	4·48	−0 4 31	298 0 29	298 0 25	50 58 18	2852·0

Remarks on the foregoing tables.

It will be observed that when two or more independent determinations have been made the wave-lengths found are often identical, rarely differ by more than ·2 of a tenth-metre. This seems to make the probable error in most cases very small—smaller than we venture to think it really may be, for there are one or two sources of error which are quite sufficient to account for a variation of ·2 of a tenth-metre in the wave-length. First the sliding tubes of the telescope, for it has a draw tube as well as the usual rack and pinion arrangement for focussing, in order to allow for the great variations in the focal distance of the uncorrected quartz lenses, and such sliding tubes have always some play, so that in the operation of reversing the photographic plate by turning the sliding tube about its axis there might easily be a small displacement.

Indeed, the measurement of plates taken successively without movement of any part of the apparatus, except the photographic slide, showed that there was such a displacement of the axis, and that it might make an error of ±10″ in the measurement of the angle, or ±·13 on an average in the value of the wave-length.

Another source of error is the want of sharp definition of some of the lines. Some of the lines are really diffuse, and in every case, as already observed, if the image is correctly focussed on the plate when the telescope is on one side of the collimator, it is always a little out of focus when the telescope is moved round to the other side. Hence there may easily be an error in measuring the distance between the lines which may easily amount to ±8″ of angular measure, or ±·1 in the value of the wave-length.

Still we do not think the probable error exceeds ±·25 of a tenth-metre.

Determination of the intermediate lines.

The wave-lengths of a sufficient number of lines of reference having been measured by the grating, the intermediate lines have been mapped by means of prisms. For this part of the work a calcite prism of 30°, cut so that one face is perpendicular to the axis of the crystal, was fixed to the end of the collimator, and a similar prism to the end of the telescope, while between them another calcite prism of 60°, cut so that the faces are equally inclined to the axis of the crystal, was maintained by a simple system of linkage at the position of minimum deviation, which is also that of single refraction. Photographs were taken at short intervals all down the scale of the spectrum of the arc and spark simultaneously. The image of the arc was focussed on the slit by the quartz lens already mentioned, and thrown just under the centre. At the same time the spark was made to pass horizontally close in front of the slit, without the interposition of a lens, but just above the centre. In this way two images were impressed on the plate overlapping one another in the middle. The distances between the lines was afterwards measured under the microscope, and the inverse squares of the wave-lengths of the intermediate lines deduced by the graphic method of interpolation between those of the lines of reference. A table of inverse squares was used for the reduction to simple wave-lengths.

The following table gives the results. In many cases there is much difference in the relative intensities of the same line in the arc and spark, and in some cases lines are visible in the photograph of the arc which are not in that of the spark, and *vice versâ*. Beyond the wave-length 2327 no spark lines seem to have made any impression on the plates, but the arc lines continue with, however, a sensible falling off in intensity up to the end of the region observed. For this region the copper lines of reference were used, as already explained, and for the highest part of it, above wave-length 2230, quartz prisms were substituted for those of calcite with advantage as regards the amount of light transmitted, but with some loss of dispersion and more of definition.

In the following table the second column gives the wave-length, and the lines marked with a "*c*" are probably carbon lines, the first gives approximately the relative intensities with which the lines are impressed on the photographic plates, 1 representing the strongest and 6 the weakest lines. It also indicates whether the line is an arc or a spark-line, *a* indicating arc-lines; *s*, spark-lines. Thus 3 *a* 1 *s* after a line indicates that its intensity in arc is 3, in spark, 1; 6 *a* indicates a line which is in arc only and of intensity 6; 2 *a s* indicates a line of intensity 2 in both arc and spark.

LIST of ultra-violet iron lines.

6 *a*	2167 4	4 *a*	2280 0	6 *a*	2341·2	2 *a s*	2384·2	2 *s*	2427 9
6 *a*	2171 7	6 *a*	2281·8	6 *s*	2341 6	6 *a* 5 *s*	2384 8	6 *a s*	2428·5
6 *a*	2173 4	6 *a*	2282 8	6 *s*	2341·8	6 *a*	2385 8	6 *s*	2428·7
6 *a*	2177 0	5 *a*	2283·0	1 *a s*	2343 1	4 *s*	2386 3	6 *a s*	2429·0
6 *a*	2178 0	5 *a*	2283 2	6 *a* 5 *s*	2343·6	6 *a* 4 *s*	2387·2	3 *a* 2 *s*	2429·7
6 *a*	2181 5	4 *a*	2283·6	4 *a* 3 *s*	2343·9	6 *s*	2388·0	6 *a s*	2430 5
6 *a*	2183 7	3 *a*	2284·0	4 *a*	2344 7	1 *a s*	2388·4	6 *a*	2430 7
6 *a*	2186 1	3 *a*	2287·1	3 *s*	2344·9	6 *s*	2389 2	6 *a* 2 *s*	2431·8
6 *a*	2186 8	3 *a*	2287·4	6 *a s*	2345·9	3 *a*	2389·9	2 *s*	2432·5
6 *a*	2191 3	6 *a*	2287·9	6 *s*	2346 4	5 *s*	2390·1	4 *s*	2433 2
5 *a*	2195·5	3 *a*	2288 8	1 *a s*	2347 8	6 *s*	2390 7	6 *a s*	2433 9
5 *a*	2199·3	5 *a*	2289·9	1 *a s*	2348·0	3 *a* 4 *s*	2391 3	6 *a* 3 *s*	2434·3
6 *a*	2200·0	4 *a*	2290 3	6 *s*	2349 0	6 *a s*	2392 4	5 *a* 3 *s*	2434·7
6 *a*	2200·2	6 *a*	2290 6	6 *a*	2349 5	6 *a*	2392 8	6 *a s*	2435 6
4 *a*	2207·5	4 *a*	2290 9	6 *a s*	2349 9	6 *a*	2394·1	6 *a s*	2436 0
4 *a*	2210 4	6 *a*	2291·4	6 *a* 2 *s*	2350·9	6 *s*	2394 7	5 *s*	2436·4
6 *a*	2211 4	3 *a*	2292·3	6 *a s*	2351 5	3 *a s*	2395 2	5 *s*	2436 9
4 *a*	2214·1	3 *a*	2293·6	6 *s*	2352 1	1 *a s*	2395·4	6 *s*	2437·3
3 *a*	2216·2	3 *a*	2294 2	5 *s*	2353·3	5 *s*	2396·5	4 *a*	2437·9
4 *a*	2225 2	3 *a*	2296 8	6 *a* 3 *s*	2354·1	6 *a*	2398 0	2 *s*	2439 0
3 *a*	2227 3	3 *a*	2297·6	3 *a s*	2354 6	6 *s*	2398·5	2 *a* 6 *s*	2439 4
3 *a*	2229 7	3 *a*	2298·0	5 *s*	2354·8	1 *a s*	2399·0	3 *a*	2439 8
6 *a*	2230 9	4 *a*	2298 6	6 *a* 5 *s*	2355·1	6 *a* 2 *s*	2400 0	3 *s*	2440·1
6 *a*	2240·2	3 *a*	2299·0	6 *a*	2355 6	6 *a s*	2401·0	6 *s*	2441·0
6 *a*	2242·2	6 *a*	2299·2	4 *s*	2356·7	6 *a*	2401·4	6 *a*	2441·5
6 *a*	2243·9	3 *a*	2300 0	1 *a s*	2358 7	6 *a* 5 *s*	2401·9	2 *a* 5 *s*	2442·3
5 *a*	2245 3	5 *a*	2300 4	6 *a s*	2359 2	6 *a* 5 *s*	2402·3	3 *a* 6 *s*	2443·7
4 *a*	2248 5	4 *a*	2301 0	2 *a* 1 *s*	2359·7	3 *a s*	2404·2	5 *a* 1 *s*	2444 3
4 *a*	2248 8	3 *a*	2301 4	2 *a* 1 *s*	2359·9	1 *a s*	2404·5	6 *a s*	2444·9
4 *a*	2250·5	3 *a*	2303·2	6 *a*	2360·3	6 *s*	2405·5	6 *a* 3 *s*	2445·4
6 *a*	2250 6	3 *a*	2303·4	6 *s*	2361·3	1 *a s*	2406·3	6 *s*	2445·9
6 *a*	2251·2	5 *a*	2304·4	4 *a* 3 *s*	2361·6	6 *s*	2406·6	6 *a* 3 *s*	2446·3
6 *a*	2251·6	5 *a*	2305 8	6 *s*	2362·9	6 *a*	2406·9	3 *s*	2447·1
4 *a*	2252 8	4 *a*	2306·0	3 *s*	2363·3	6 *a*	2407·3	3 *a s*	2447·5
4 *a*	2255 4	2 *a*	2308 6	3 *s*	2363·5	6 *a s*	2407·6	6 *a*	2448·1
4 *a*	2259·2	6 *a*	2309 8	2 *a* 1 *s*	2364 4	6 *s*	2408·4	6 *a*	2448·5
4 *a*	2259 8	6 *a*	2310·6	6 *a*	2365 1	1 *a s*	2410·2	4 *s*	2449·6
5 *a*	2260·4	5 *a*	2311·0	5 *s*	2365·3	1 *a s*	2410·7	6 *a* 4 *s*	2450·0
5 *a*	2260 7	6 *a*	2311·6	3 *a s*	2366·2	6 *a*	2411 4	6 *a*	2450·7
6 *a*	2262 4	6 *a*	2312·0	2 *a* 1 *s*	2368·2	1 *a s*	2413·0	6 *a* 5 *s*	2451·0
6 *a*	2262 8	2 *a*	2312·7	4 *a*	2369 1	6 *a s*	2413·8	6 *a*	2451·3
6 *a*	2263·2	6 *a*	2313 6	5 *s*	2369·6	6 *a s*	2414·8	6 *a*	2451·8
5 *a*	2264·2	6 *a*	2316·7	4 *a s*	2370·1	6 *a*	2415·4	6 *a*	2452·3
5 *a*	2264 7	5 *a*	2317·5	4 *a*	2371·1	6 *a* 5 *s*	2416·3	6 *s*	2452·9
6 *a*	2265·7	6 *a*	2317·7	4 *s*	2372·3	6 *a*	2417·1	3 *a*	2453·2
6 *a*	2266 6	5 *a*	2319·2	6 *a*	2372·7	6 *a* 2 *s*	2417·5	5 *s*	2453·5
5 *a*	2266·8	6 *a*	2319 6	4 *s*	2373·3	6 *a* 5 *s*	2418 2	6 *s*	2453·8
3 *a*	2267·2	3 *a*	2319·9	2 *a* 1 *s*	2373·4	3 *a*	2418·9	3 *s*	2454·3
6 *a*	2268 8	2 *a* 1 *s*	2326·9	6 *a*	2374·1	6 *a*	2419·4	6 *a*	2455·3
4 *a*	2270 5	6 *a*	2329·3	2 *a* 1 *s*	2374·9	6 *s*	2419 7	6 *s*	2455·7
4 *a*	2271·5	2 *a* 1 *s*	2330 9	6 *a* 3 *s*	2376·2	6 *a*	2420·0	6 *a*	2456·0
4 *a*	2271·8	2 *a* 1 *s*	2332 5	6 *a*	2376·9	6 *a*	2420 7	6 *s*	2456·4
4 *a*	2272·5	6 *a*	2333·1	6 *a*	2377·6	6 *a*	2421 3	2 *a* 5 *s*	2457·4
4 *a*	2273·8	6 *a*	2334 2	6 *s*	2378·2	6 *a* 3 *s*	2422·4	6 *a*	2458·2
4 *a*	2274·9	6 *a*	2334·5	6 *s*	2378·8	6 *a* 4 *s*	2422·9	6 *a* 1 *s*	2458·5
6 *a*	2275·2	6 *a*	2334·8	2 *a* 1 *s*	2379·0	5 *a* 1 *s*	2423·8	6 *a* 4 *s*	2460·2
4 *a*	2275·7	2 *a* 1 *s*	2337 7	2 *a* 1 *s*	2380·5	6 *s*	2424·3	6 *a*	2460·8
4 *a*	2276·9	2 *a* 6 *s*	2339·0	1 *a s*	2381·7	6 *a s*	2425·0	6 *a* 3 *s*	2461·0
4 *a*	2277·5	3 *a*	2339·3	3 *a s*	2382·7	6 *a* 5 *s*	2425·4	3 *s*	2461·4
3 *a*	2279·7	5 *a* 6 *s*	2340·0	2 *a s*	2383·0	6 *s*	2427·0	3 *a*	2461·9

List of ultra-violet iron lines—(continued).

2 *a* 6 *s*	2462·3	6 *s*	2507 9	5 *s*	2554·8	6 *a*	2608·2	6 *a*	2672·4
6 *a* 4 *s*	2462·8	5 *a*	2508 5	6 *a*	2554·9	6 *a s*	2608·7	6 *a*	2674 6
5 *a*	2463·4	5 *s*	*c* 2508·8	5 *s*	2555 2	6 *a*	2609·1	6 *a*	2675·1
4 *s*	2463·7	2 *a* 5 *s*	2510·6	6 *a*	2556 0	6 *s*	2609 3	6 *s*	2676 1
4 *s*	2464·5	6 *a* 1 *s*	*c* 2511·4	6 *a*	2556 6	6 *a*	2610 3	6 *a*	2677·2
3 *a* 4 *s*	2464·7	6 *a*	2511·6	6 *s*	2557·2	6 *a s*	2610·7	2 *a* 6 *s*	2678·5
3 *s*	2465·4	3 *a*	2512·0	6 *a*	2558·3	1 *a s*	2611·4	6 *a*	2679 9
5 *a* 2 *s*	2466 4	3 *a* 4 *s*	2512·2	6 *s*	2558·9	6 *a*	2612·3	5 *s*	2680 4
5 *a*	2467 2	6 *a*	2513·2	5 *s*	2559·6	1 *a s*	2613 8	6 *a*	2680 8
5 *s*	2467 8	3 *a s*	*c* 2514·1	6 *a* 4 *s*	2560·0	6 *a*	2614·0	6 *a*	2681 5
3 *a* 6 *s*	2468·4	6 *a*	2514·3	6 *a*	2560 3	6 *a*	2615 0	5 *s*	2682 0
3 *s*	2469·0	4 *s*	2514·7	6 *a*	2560·9	1 *a s*	2617·2	5 *s*	2682·4
2 *s*	2470·3	1 *a* 6 *s*	*c* 2515 8	6 *a*	2561·5	6 *a*	2617·6	6 *a*	2683·5
6 *a*	2470·5	6 *a*	2516·3	6 *a*	2561·9	6 *a*	2618·3	6 *a* 1 *s*	2684 2
3 *a* 4 *s*	2471·9	6 *a* 3 *s*	2516 8	2 *a s*	2562 3	4 *s*	2618 6	6 *s*	2685 7
2 *a* 5 *s*	2472·4	5 *a*	2517·4	2 *a s*	2563·2	5 *a s*	2619·9	6 *a*	2686·0
6 *a*	2472·7	3 *a* 6 *s*	2517·8	6 *a*	2564·2	5 *s*	2620 4	6 *a*	2686·8
5 *s*	2472·9	6 *a*	2518·5	6 *a*	2565·1	3 *a s*	2621 2	6 *a*	2687 3
3 *a s*	2474·5	3 *a s*	*c* 2518·8	4 *s*	2566 0	6 *s*	2622 6	3 *a s*	2688·8
6 *s*	2474·9	5 *a* 6 *s*	2519 3	3 *a s*	2566·7	3 *a* 6 *s*	2623·1	5 *a*	5689·3
6 *s*	2475·5	6 *a* 3 *s*	2520 8	5 *s*	2568·1	6 *a*	2623 6	5 *a*	2689·5
6 *a*	2475·8	6 *a* 3 *s*	2521 5	6 *a s*	2568 6	1 *a s*	2625·2	6 *a*	2690 9
6 *s*	2476 0	1 *a* 2 *s*	2522·5	6 *a s*	2569 4	6 *a s*	2626·2	6 *s*	2691 2
5 *a* 6 *s*	2476·5	5 *a* 6 *s*	2523·3	5 *a*	2570·1	6 *a*	2626·8	6 *a*	2691·7
5 *s*	2477·1	2 *a* 6 *s*	*c* 2523·9	4 *s*	2570·6	1 *a s*	2627·9	6 *a* 1 *s*	2692 1
6 *s*	2477 9	5 *a*	2524·7	6 *s*	2571·2	6 *a* 5 *s*	2629·2	6 *s*	2693 4
1 *a* 3 *s*	2478·3	6 *a* 2 *s*	2525·1	6 *a*	2572 5	6 *a* 5 *s*	2629·7	5 *a*	2694·0
6 *s*	2479·0	5 *a* 2 *s*	2526 0	6 *s*	2572·8	1 *a s*	2630 7	6 *a*	2694·4
6 *a*	2479·2	5 *s*	2526 7	6 *a* 3 *s*	2574·0	1 *a s*	2631 0	6 *s*	2694 7
2 *a* 6 *s*	2479·5	2 *a* 5 *s*	2527·1	6 *a*	2574·8	5 *a* 6 *s*	2632 0	5 *a*	2695·0
6 *a* 2 *s*	2480·0	6 *a*	2527·9	5 *a*	2575·3	6 *a*	2632·3	4 *a*	2695·6
6 *s*	2480·7	6 *a*	*c* 2528·1	4 *a* 6 *s*	2575 7	5 *s*	2632 9	6 *a*	2695·9
6 *s*	2481·3	3 *a* 6 *s*	2528·9	3 *a*	2576 2	6 *s*	2635·1	6 *a*	2696·6
6 *a* 3 *s*	2481·8	2 *s*	2529·2	4 *s*	2576·5	3 *a* 6 *s*	2635·5	4 *s*	2697 0
3 *s*	2482·4	3 *a*	2529·6	3 *a s*	2577·4	6 *a s*	2636·1	6 *a*	2697·7
1 *a* 6 *s*	2482·9	6 *s*	2529·9	6 *a*	2578·3	6 *a*	2636·6	4 *a* 5 *s*	2698·6
6 *s*	2483·3	4 *a* 6 *s*	2530·4	6 *a*	2578 7	3 *s*	2637·3	6 *s*	2699·8
3 *a s*	2483·7	6 *a*	2531·1	6 *s*	2578·9	4 *s*	2639·2	6 *a* 6 *s*	2701 2
6 *a*	2484·7	5 *a* 6 *s*	2532 0	6 *a*	2579·3	6 *s*	2640·7	6 *a*	2702·6
6 *a*	2485·7	6 *a*	2532·4	6 *a*	2579·5	4 *a* 6 *s*	2641·4	1 *s*	2703 6
5 *a* 2 *s*	2486 1	6 *a*	2532 6	6 *a*	2579 9	6 *s*	2641·7	3 *a*	2705 6
5 *a*	2486·4	3 *a* 2 *s*	2533·4	6 *a*	2580 3	3 *a* 6 *s*	2643 8	1 *a* 3 *s*	2706·0
5 *a*	2486 8	6 *a* 2 *s*	2534 2	6 *s*	2580 6	6 *a* 3 *s*	2644·9	6 *a* 4 *s*	2706 7
5 *a*	2487·1	2 *a* 3 *s*	2535·2	6 *a*	2580 9	5 *a*	2645·2	3 *a*	2708·1
1 *a* 3 *s*	2487·7	5 *a* 1 *s*	2536·6	4 *a*	2581 7	6 *s*	2645 8	4 *s*	2708·7
6 *a*	2488·7	3 *a*	2536·9	4 *a* 3 *s*	2582 0	4 *a* 6 *s*	2647·3	6 *a s*	2709·7
4 *s*	2489·2	4 *s*	2538 0	2 *a* 6 *s*	2584·0	4 *s*	2649 2	4 *a* 6 *s*	2710·1
1 *a* 2 *s*	2489 5	5 *a* 1 *s*	2538·6	1 *a s*	2585 4	6 *a* 5 *s*	2650·4	3 *a* 4 *s*	2711 2
1 *a* 2 *s*	2490·5	6 *a*	2539·1	3 *a s*	2587 5	6 *a*	2650·9	3 *s*	2711·5
2 *a*	2491 0	5 *s*	2540 4	6 *s*	2588 2	6 *s*	2652 2	6 *s*	2711 9
3 *s*	2491·1	2 *a* 4 *s*	2540·8	6 *s*	2590 0	6 *s*	2653·3	6 *a*	2713·5
6 *a s*	2492·0	4 *s*	*c* 2541·6	3 *a s*	2591·0	6 *s*	2654·4	1 *a s*	2713·8
3 *a* 1 *s*	2492·9	3 *a* 6 *s*	2541 7	6 *a*	2591·7	5 *a s*	2655·7	6 *a*	2714·4
5 *a s*	2493·7	6 *s*	2542·4	3 *s*	2592·2	6 *a*	2656 4	6 *a*	2714·9
6 *a*	2493·9	6 *a* 3 *s*	2543 0	3 *a s*	2593 1	6 *a* 3 *s*	2657·3	6 *a* 3 *s*	2715·7
5 *a* 4 *s*	2495·6	3 *a* 6 *s*	2543·7	6 *a*	2593·5	5 *a*	2660·8	6 *a*	2717·4
3 *a* 6 *s*	2496·3	4 *a*	2544·5	6 *s*	2594·5	4 *a* 6 *s*	2661·6	2 *a* 6 *s*	2718·0
6 *a* 2 *s*	2497·5	5 *s*	2544 9	6 *a*	2595·2	6 *s*	2662·2	1 *a* 3 *s*	2718 5
6 *a* 1 *s*	2498 7	2 *a* 6 *s*	2545·8	6 *a*	2596·0	5 *a*	2663·5	5 *a*	2719 7
4 *s*	2500·7	5 *a s*	2546·6	1 *a s*	2597·8	6 *a*	2664·0	1 *a* 4 *s*	2720 3
2 *a* 6 *s*	2500·9	6 *s*	2547·0	1 *a s*	2598·9	6 *a* 1 *s*	2664 2	6 *s*	2721·5
5 *a*	2501·4	6 *a*	2547·8	6 *a*	2599 7	5 *a*	2665·7	6 *s*	2721 7
6 *a* 2 *s*	2502·1	5 *s*	2548·4	6 *a*	2603·5	1 *a s*	2666 1	5 *s*	2722 3
6 *a*	2503·0	5 *s*	2549·0	6 *s*	2603·8	6 *s*	2666·7	1 *a* 4 *s*	2723·1
3 *s*	2503 1	5 *s*	2549·1	6 *a*	2604·4	6 *a*	2667·2	5 *a* 3 *s*	2724·3
3 *s*	2503·6	2 *a*	2549·2	5 *s*	2604 9	6 *s*	2668·5	5 *a*	2725·5
6 *a s*	2504·9	5 *s*	2549·7	5 *s*	2605·1	6 *a*	2668·7	6 *s*	2726 0
6 *a*	2505·2	5 *s*	2550·3	3 *a*	2605·3	6 *s*	2669·2	4 *a* 1 *s*	2727·1
3 *s*	2505·8	6 *a s*	2550 8	6 *s*	2605·6	6 *s*	2669 7	6 *a*	2727·5
6 *a*	2506·2	6 *a*	2552·3	5 *s*	2606·1	6 *a*	2669 9	6 *a* 5 *s*	2728·3
3 *a* 6 *s*	*c* 2506·6	6 *a s*	2552·8	6 *a s*	2606·5	6 *s*	2670·8	6 *s*	2729·1
3 *a* 6 *s*	2507·6	6 *s*	2553·4	3 *a s*	2606·7	6 *a* 5 *s*	2671 8	5 *a* 3 *s*	2730·2

LIST of ultra-violet iron lines—(continued).

6 s	2731·5	6 s	2764·7	6 a s	2803 2	6 a 4 s	2848 0	5 a s	2894·0
6 s	2732·5	6 s	2765·3	6 s	2803·8	6 a s	2848·2	5 a 4 s	2894·5
2 a 4 s	2733 1	6 a	2766·8	6 a 5 s	2804·2	6 s	2849·3	5 s	2896 7
6 a	2733·7	3 a 1 s	2767·2	6 s	2804·9	6 s	2855·3	6 a	2897·8
6 a	2733 9	6 a 4 s	2768·8	6 s	2805·4	6 s	2856·7	5 a	2898 9
3 a 5 s	2735·0	6 a 4 s	2769·1	3 a 6 s	2806·7	4 s	2857·9	5 a	2900·8
5 a 1 s	2736·5	6 a	2769·4	6 a	2807·9	6 a	2858·3	5 a	2901 3
3 a 6 s	2736 9	6 a s	2770 3	6 a s	2809·7	6 s	2860 9	6 s	2902 1
1 a s	2739 1	6 s	2771 1	6 s	2810 9	6 s	2862·1	6 a	2903·5
5 s	2741 1	2 a 5 s	2771 9	6 a	2811·7	6 a	2862·4	6 a s	2905·8
1 a 5 s	2742·0	6 a	2773·1	6 s	2812·2	4 a	2863·1	6 a s	2907·1
3 a 2 s	2742 8	6 a 5 s	2774 5	2 a 4 s	2812 8	4 a	2863·6	6 a	2908 2
6 a	2743·3	6 s	2776·1	6 s	2813 4	6 s	2864·7	6 a	2908·9
3 a 5 s	2743 7	6 s	2776 9	6 a	2815·1	5 a	2866·2	6 s	2910·5
5 a	2744 2	6 s	2777·7	6 a s	2817 0	6 s	2866·5	1 a 6 s	2911·5
2 a 3 s	2746·1	4 a 5 s	2777 9	6 a s	2819 0	6 a	2867·1	6 a	2913 6
2 a s	2746·6	6 a	2778 3	6 a	2820·4	6 a	2868·0	5 a 6 s	2917·4
1 a s	2749 0	6 a 2 s	2778·9	3 a 6 s	2822·9	3 a 6 s	2869·0	6 a	2920·0
2 a 5 s	2749 8	5 a 6 s	2781·6	6 a	2823 9	6 s	2870 7	6 s	2921 5
6 a	2750 6	1 s	2783·4	3 a 6 s	2825 1	4 a 5 s	2872 0	6 a s	2922 8
5 s	2750·8	6 a	2784·2	6 s	2827·0	4 s	2873 0	6 a	2923 2
6 s	2752·1	3 s	2785 1	6 a s	2827 3	3 a	2873 6	6 a	2924·7
6 a 1 s	2753 0	1 a 3 s	2788 0	6 a 5 s	2828 3	6 a 5 s	2874 9	6 a	2925·2
6 a	2753 5	6 a	2789·5	2 s	2831 0	6 s	2876·4	6 a 2 s	2926 0
6 a	2753 9	6 s	2790 3	3 a 6 s	2831·8	4 a	2876·8	1 a 4 s	2928 3
6 a	2754 3	6 a	2791·5	5 a	2832 4	4 a	2878 2	6 s	2931 1
1 a s	2755 5	6 a	2792·2	5 a	2832 8	6 a 5 s	2880 4	5 a 6 s	2932 4
3 a	2756 2	4 s	2793 3	3 s	2835·2	6 a 4 s	2883·3	1 a 5 s	2936 4
6 s	2756·9	3 a	2794 5	6 s	2836 7	6 s	2885·5	5 a	2937 3
4 a	2757·2	6 s	2796·3	5 a 6 s	2837·7	6 a	2885 8	4 a 6 s	2938 7
6 a	2759 7	5 a s	2797·4	6 a	2839·6	6 a	2887·3	6 a	2939 9
3 a s	2761 7	3 a	2797 9	6 a 3 s	2840·3	6 s	2887·6	2 a	2940·8
4 a	2761 9	6 a 4 s	2798 8	6 a s	2843 1	6 a	2889 2	6 a	2943·1
6 s	2762·4	6 a	2799 4	2 a 4 s	2843·6	6 a	2891 2	6 a 2 s	2944·0
4 a	2763·0	6 a	2800 1	5 a 4 s	2845 3	6 a	2892·0	6 a	2944·6
6 s	2763·6	3 a	2800 8	6 a	2846·5	6 a	2893 2	U	2947 3
6 a	2764 0	6 a	2801 8						

The accompanying map is drawn to a scale double that of ÅNGSTRÖM's and CORNU's maps of the solar spectrum. Those lines which are common to both arc and spark are drawn right across from top to bottom, while those which are in the arc only are not continued to the bottom, and those which are in the spark only do not begin at the top; so that the upper portion of the map represents the arc spectrum, the lower the spark spectrum, and they overlap in the middle.

We do not pretend to say that every line in this map really belongs to iron, for commercial iron wire was used to produce it, and the map may therefore probably include lines of a good many metals, certainly manganese lines in the arc, but it will not the less serve the purpose for which it has been made, namely: for reference in determining the approximate wave-lengths of lines of any spectra.

PART II.

(Received June 15, 1882.)

The account of the ultra-violet spectra of fifteen metals here recorded is a first instalment of the results of observations which we have accumulated during the past three years, but have not heretofore been able to reduce. During that time we have taken some thousands of photographs of the electric arc under various conditions, and especially in crucibles of lime and magnesia (as previously described by us), and in the presence of most of the known metals; but with the exception of CORNU's map of the ultra-violet solar spectrum giving the chief iron lines and a few of those of other elements, up to the line U we have had little to aid us in the exploration of a new field and the assignment of the several lines to the elements producing them, and the measurement of our many photographs has cost both time and patience. Dr. W. A. MILLER long ago published an account of his photographs of the spark spectra of the elements, and Mr. HARTLEY has recently (Trans. Roy. Dublin Soc.) published photographs of the spark spectra of several elements which are a great improvement on those previously published. But those give spark spectra only, are taken with an apparatus of small dispersion, and are not reduced to scale, so that they give qualitative rather than quantitative results. The spectra which we here describe are those of the arc up to the wave-length 2200, and we give in each case the approximate wave-lengths of the lines observed. For some few of the lines of tin and aluminium the wave-lengths have been determined by means of a grating as described in the first part of this paper, but in all other cases they have been derived by interpolation from the wave-lengths of the neighbouring iron lines. In the map which accompanies this paper we have given in the top line the principal lines of iron for convenience of reference, and in the lowest line the arc lines of carbon with which it is necessary to be acquainted as they are always present, though varying much in intensity, in the arc taken between carbon electrodes. The scale of this map is one-half that of ÅNGSTROM's "Normal Solar Spectrum."

We have already, in describing the visible spectra of the alkali metals and that of magnesium, called attention to probable harmonic relations between the lines. This relation manifests itself in three ways—first, by the repetition of similar groups of lines; secondly, by a law of sequence in distance, producing a diminishing distance between successive repetitions of the same group as they decrease in wave-length; and thirdly, a law of sequence as regards quality, an alternation of sharper and more diffuse groups, with a gradually increasing diffuseness and diminishing intensity of all the related groups as the wave-length diminishes.

The first relationship has long since been noticed in the case of the sodium lines which recur in pairs, and we have observed that the potassium lines between the extreme red and violet pairs are repetitions of a quadruple group, while the lithium lines (with the exception of the blue line mentioned below) are single, and one set of those of magnesium triplets. We now record a second harmonic* series of potassium lines which appear to be pairs, and the violet pair, and possibly the red pair too, belong to this series. Lithium shows a second harmonic series of single lines high up on the scale. Calcium gives a long series of well marked triplets; zinc likewise gives a series of triplets; aluminium gives pairs, and in the highest region triplets; thallium gives a series which seem to be quadruple groups with two of the four lines in each of much greater intensity than the rest. The alternations of sharper and more diffuse groups are generally apparent and are very marked in the cases of calcium and zinc. The diminishing distance and intensity and increasing diffuseness of successive repetitions of the same group as the wave-length diminishes, are in all the cases mentioned very plain. In all these cases the different lines forming a group are tolerably close to one another, so that successive repetitions of a group do not overlap one another, but it may be that in other cases the lines forming one group may be so far apart that the most refrangible line of one group may be more refrangible than the least refrangible line of the next repetition of the group; the groups and their sequence will thus be much less easily recognised.

Potassium.

The ultra-violet spectrum of potassium, so far as we have observed it, is apparently one harmonically related series of which the first member above the visible spectrum is a double line just below the solar line O; the next falls between Q and R, and the others follow at decreasing intervals, the seventh and last that we have observed falling just above U. It is only in the case of the line near O that we have been able to make sure that it consists of a pair of lines, but it is very probable that all are pairs in reality; all are strongly reversed, as might be expected from the volatility of the metal, and expanded when a fresh quantity of the metal or its compounds is introduced into the arc, so that the separation of the pairs, if such they be, could not be seen, while the more refrangible lines die away and are not recognisable as *bright* lines amongst the many lines which come out in the arc, as the alkali metal is dissipated. The line between Q and R, which is a strong line, happens to be in a region where the lines of iron, manganese, and chromium lie very closely, so that we cannot pronounce with certainty that it is a double line.

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* By an "harmonic series" of lines we merely mean a series of overtones of a fundamental vibration we do not mean that they follow the simple arithmetical law of an ordinary harmonic progression, but are comparable rather with the overtones of a bar or bell than with those of a uniform stretched string.

POTASSIUM lines.

Approximate wave-length.	Remarks.
3445·0 } 3443·6 }	Double line.
3216·5	All are easily reversed.
3101·0	
3033·0	
2992·0	The lines become weaker as they are more refrangible.
2963·4	
2942·0	

Sodium.

The sodium lines observed by us also form one apparently harmonic series with the double line, wave-length 3301, observed by CORNU. In this case also we have not been able to make out that any of the lines above 3301 are double, as when there is enough sodium present to develop them decidedly they are always more or less diffuse and reversed. Indeed, the line at 3301 is a very close pair and it is not often seen as two lines.

One line is so near to the very strong magnesium line, wave-length 2852, that the apparent development of the magnesium line by sodium was for some time an enigma to us. The sodium line is a little less refrangible than the magnesium line.

SODIUM lines.

Approximate wave-length.	Remarks.
3301·0	CORNU's double line.
2853·3	All the lines are easily reversed.
2679·0	
2593·3	

Lithium.

We have already described one apparently harmonic series of lithium lines extending into the ultra-violet up to about wave-length 3799. This series we described as all single lines though alternately sharp and diffuse. This description is correct, except that we have since found that one line of the series, namely, the strong blue line at wave-length 4604, is really a double line. When a fresh dose of lithium, or of some one of its compounds, is introduced into the arc, a second weaker line comes out on the more refrangible side of the strong blue line, and gives to it all the appearance of a reversal with an expansion of the bright wings unequally extended on the two sides. As the strong blue line is, however, often really reversed, the effect is then that of a double reversal, that is to say, the appearance is that of a broad bright band with a narrower dark band within it and a bright line in the middle of the dark band. The second line rather quickly dies out as the lithium evaporates, leaving the strong blue line comparatively permanent. We have never observed any such second line, or companion, to any of the other lines of lithium. The new series begins with a line at wave-length about 3232, and dies out with a very diffuse line at about wave-length 2359. The following is a list of the ultra-violet lines we have observed.

Approximate wave-length.	Remarks	Approximate wave-length.	Remarks
3984·5		2561·5	Reversed.
3913·5	Diffuse.	2475·0	„
3862·3	Somewhat obscured by the cyanogen bands in this region	2425·5	
3799·0		2394·5	Diffuse.
3232·0		2373·5	Very diffuse
2741·0	Reversed.	2359·0	Very diffuse and weak.

Barium.

The barium lines are numerous, but do not fall into easily recognised harmonic series.

BARIUM lines.

Approximate wave-length	Remarks.	Approximate wave-length.	Remarks.
3991·8		3320·9	
3908·5		3279·8	
3891·0		3261·0	
3793·5		3070·3	
3660·7		2785·1	
3598·7		2771·0	
3592·8		2739·0	
3579·1		2702·0	
3544·0		2647·0	
3524·5		2634·5	
3499·2	Very strong, reversed.	2596·7	
3419·3		2542·7	
3375·6		2347·0	Strong.
3354·8		2335·0	Very strong.
3347·7		2304·5	Strong.

Strontium.

Strontium, and its compounds, produce a line at wave-length 3705 coincident, or nearly so, with one of the lines ascribed to calcium by CORNU. We have so often observed this line much reinforced by strontium without any increase of the other calcium lines which are always present in the arc from calcium in the carbon electrodes, that we think we are justified in putting down a strontium line at this place. Two other lines of this metal are close to, but not coincident with lines of barium.

STRONTIUM lines.

Approximate wave length.	Remarks.	Approximate wave-length.	Remarks
3705·0	Coincident or nearly so with a calcium line.	3458·0	Diffuse
		3379·5	
3653·0		3364·8	
3547·0		3305·2	
3527·0		2931·1	
3498·0			
3464·0			

Calcium.

CORNU has mapped two calcium lines, one on either side of the solar line M, and four other lines, of which one is coincident with the solar line R, one slightly less refrangible, and the other two more refrangible. One of these lines, at wave-length 3168·5, we have never certainly seen, but the others are well developed when calcium compounds are put into the arc. Besides these we have always seen when calcium, or one of its compounds, is present in moderate quantity a series of triplets analogous to those of magnesium. Each triplet consists of two strong lines with a rather weaker line on their more refrangible side. This series appears to be harmonically related to the well-known blue triplet at wave-length 4454–24. The first repetition of this triplet occurs close to H, one line of the triplet falling below H, while the other two lines fall between H and K. The next triplet falls between N and M, and the next between O and N, and so on at decreasing intervals, the most refrangible repetitions becoming very faint and diffuse, so that in the last, a little below S, we have only been able to distinguish the strongest two lines of the triplet. The triplets are alternately diffuse and sharp, those near H, between O and N, and so on alternately, being the sharper. The diffuse triplets are stronger than the others and more easily reversed. Beyond this series we have noticed only one calcium line, and that is high up on the scale, at wave-length about 2398.

CALCIUM lines.

Approximate wave-lengths.	Remarks.	Approximate wave-lengths.	Remarks.
3967·7	H.	3285·0	
3972·3		3273·5	Sharp triplet.
3956·0	Sharp triplet.	3268·5	
3947·9		3224·5	
3933·0	K.	3213·0	Diffuse triplet.
3736·4	Mapped by CORNU.	3208·0	
3705·5	Ditto.	3181·0	Mapped by CORNU.
3644·0	Very strong.	3179·0	R. ditto.
3631·0	Readily reversed.	3168·5	Ditto, not seen by us.
3623·5		3158·8	Mapped by CORNU.
3486·5		3151·0	
3474·5	Sharp triplet.	3141·0	
3468·0		3136·0	
3359·5		3117·5	Weak, very diffuse.
3347·5	Very strong.	3108·0	
3342·0		2398·0	

Zinc.

Zinc is another metal which gives a well-marked apparently harmonic series of triplets, but the different lines of each group are further separated than in the calcium or magnesium triplets. The middle line of the first triplets confounds with the sodium pair wave-length 3301, but by reason of the diffuse character of the zinc line we have not been able to decide whether the coincidence is more than approximate.

ZINC lines.

Approximate wave-lengths.	Remarks.	Approximate wave-lengths.	Remarks.
3342·0		2608·5	
3301·0		2582·0	Diffuse.
3281·0		2569·7	
3070·0		2516·0	
3035·0		2491·5	
3017·0		2480·0	
2800·0		2464·5	
2770·0	Diffuse.	2440·0	Very diffuse.
2756·0		2430·0	
2713·3			
2684·0			
2670·5			

Mercury.

As might be expected from its volatility, it is difficult to obtain lines of mercury in the arc; but one line gives a reversed image of itself at wave-length 2536·8. This line is very bright in the flame of cyanogen, containing vapour of mercury.

Gold.

Gold also gives us but few lines. The three lines we record are perhaps harmonically related.

Gold lines.

Approximate wave-length	Remarks.
3122·8	
2675·4	Reversed.
2427·5	Ditto.

Thallium.

Groups of thallium lines manifestly similar to one another recur, and are probably harmonically related. This recurrence is more evident in the photographs in which the lines which are expanded and reversed by the introduction of fresh metal are at once recognised. The pairs at 2921, 2710, 2609, and the lines at 2552, 2517 seem to fall into one series.

Thallium lines.

Approximate wave-length.	Remarks.	Approximate wave-length.	Remarks
3775·6		2714·6	Very diffuse.
3528·3		2710·4	Reversed.
3517·8		2708·8	Strong, diffuse reversed.
3228·1		2699·7	Very diffuse.
2943·9		2665·0	Diffuse.
2921·3	Very strong.	2652·3	
2917·8		2609·4	Reversed.
2895·2		2608·6	Strong, reversed.
2825·8		2552·0	Reversed.
2826·9		2517·0	Diffuse.

Aluminium.

The spectrum of aluminium is comparatively simple. The well-known pair of lines between H and K seem to be repeated twice in the region above without much, if any, diminution of strength, but we have not observed any such lengthened sequence of repetitions of these lines as we have of the lines of magnesium and other metals. Higher on the scale we come to another series of groups which are triplets, or perhaps quadruple groups, for the first and strongest group shows a faint fourth line which we have not observed in the succeeding groups. The repetitions we record are only two, but they lie in a part of the spectrum so near the limit of transparency of calcite that it is quite possible that there may be more beyond, which will show themselves when quartz prisms are used; a strong triplet near N appearing in the spark, wave-length about 3605, 3598, 3585, does not show in the arc so far as we have observed.

ALUMINIUM lines.

Approximate wave-length.	Remarks	Approximate wave length.	Remarks
2659·8	Strong, frequently reversed.	2268·7	Strong, diffuse.
2652·0		2263·1	,, ,,
2574·5		2257·3	
2567·5	,, ,, ,,	2216·0	
2378·4	A fourth faint line close to the middle line of this group. Middle line very strong, generally reversed.	2210·0	Strong, diffuse.
2373·2		2205·0	Diffuse.
2366·9			

Lead.

The lines of lead are numerous and strong, and many of them readily reversed. We have not yet traced any probably harmonic series amongst them.

Lead also gives some indefinite bands of continuous light about the region wave-length 2500.

LEAD lines.

Approximate wave length.	Remarks.	Approximate wave-length.	Remarks.
4019·0		2801·1	Often reversed.
3801·0		2721·0	
3739·3		2706·1	
3683·3	Strong, often reversed.	2697·0	Middle of a very diffuse band.
3670·7		2662·7	
3639·3	,, ,, ,,	2650·5	Diffuse.
3572·0		2627·8	
3260·0	Nearly coincident with a line of tin.	2613·7	Strong, often reversed.
3238·6		2575·7	Very diffuse.
3219·6		2476·5	Strong, reversed.
3118·5		2446·1	,, ,,
2981·0		2443·7	,, ,,
2973·5	Weak lines.	2428·5	
2967·0		2411·5	,, ,,
2872·0		2401·8	,, ,,
2850·5	A little above the magnesium line, sometimes hidden by the expansion of the latter.	2399·4	
		2393·7	Very strong, reversed.
		2388·8	
2832·9	Very strong and diffuse.	2332·0	
2822·5	Generally reversed.		

Tin.

Tin is remarkable for the number and strength of its lines in the higher region of the spectrum, while its lines of lower refrangibility are so feeble that we have never seen any in the arc in the visible part of the spectrum. CORNU has recorded one line at wave-length 3260, and we have had no difficulty in recognising this line, but a line very nearly in that place is also developed by lead when other tin lines are not developed, and we have not been able to perceive that these lines are separable in any of our photographs. Many of the higher lines of this metal are easily reversed, indeed are almost always reversed in our photographs.

TIN lines.

Approximate wave-length.	Remarks.	Approximate wave-length.	Remarks.
3326·0		2523·5	
3260·0	Given by CORNU	2495·5	
3175·0		2493·5	
3141·7		2483·1	
3033·0		2429·5	
3008·5		2421·5	
2986·4		2407·9	
13·1		2392·5	
2862·8		2364·7	
2839·5		2357·7	
2813·5		2354·5	
2812·5		2334·3	
2787·5		2317·0	
2784·7		2286·9	
2779·5		2282·5	
2761·5		2275·4	
2660·7		2251·0	Strong, reversed.
2636·5		2245·8	Strong.
2593·5		2231·3	
2571·0		2210·7	
2557·5		2198·7	
2546·1		2194·1	
2530·7			

ANTIMONY lines.

Approximate wave-length.	Remarks.	Approximate wave-length.	Remarks.
4032·0		2597·5	
3637·0	Close above a lead line.	2528·0	
3265·0		2426·0	
3230·8		2383·3	
3228·0		2313·0	Very strong, reversed.
3028·0		2310·0	
2876·5			

BISMUTH lines.

Approximate wave-length.	Remarks.	Approximate wave-length.	Remarks.
3595·3		2799·0	
3510·4		2780·0	
3396·2		2730·0	
3066·0	Very strong, often reversed.	2593·0	
3023·5		2524·0	
3000·0		2515·4	
2996·0		2448·0	
2937·4		2435·5	
2897·0		2431·0	
2862·0		2400·8	Very strong.
2810·0		2277·0	Weak, reversed.

Carbon.

In our map we give the carbon lines as developed in the arc. These occur in the arc taken between poles of purified graphite in air, and in nitrogen, and in carbonic acid gas, and they are always present in the arc taken in our crucibles. Most of them are also in the spark spectrum of carbon as described by us (Proc. Roy. Soc., xxxiii., 403), but some of the spark lines are not developed in the arc, and there are two lines in the arc which we did not notice in the spark.

CARBON lines.

Approximate wave-length.	Remarks.
2881·1	Not observed in spark.
2528·1	
2523·9	
2518·8	
2515·8	
2514·1	
2506·6	
2478·3	The strongest line
2434·8	Not observed in spark.

IV. *Experiments on the Value of the British Association Unit of Resistance.*

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PART I.

BEFORE leaving Cambridge, Professor CHRYSTAL, in the summer of 1878, wound with great care two large coils of about 50 centims. in diameter to be used for a redetermination of the value of the British Association unit of resistance. A galvanometer of special construction with double coils, one of thick wire the other of thin, was also wound by him and mounted with WEBER's suspension.

The coils were to be placed at a known distance apart, so that the coefficient of mutual induction could be calculated.

The apparatus remained unused till the spring of last year, when the experiments described in the present paper were commenced.

The method used is similar to those employed by KIRCHHOFF (Pogg. Ann., lxxvi.) and ROWLAND (American Journal of Science and Arts, vol. xv., 1878). The coefficient of mutual induction of the two coils is determined by calculation from the geometrical data. A current is passed through one coil, the other being in circuit with a ballistic galvanometer, and the induced current, produced when the primary current is broken or reversed, is measured by the throw of the galvanometer needle. The primary current itself being then measured by some method, we have enough data to determine in absolute measure the resistance of the secondary circuit and the galvanometer. But this resistance can be measured in terms of the B.A. unit, and hence a value obtained for the latter.

Professor ROWLAND measured his primary current by means of a second galvanometer, the constant of which he compared with that of the ballistic galvanometer both by direct experiment and by calculation.

In our experiments the value of the primary current was obtained by passing by means of a shunt about $\frac{1}{8000}$th part of it through the ballistic galvanometer; we thus eliminate from our equations the values of the galvanometer constants, as well as the correction which Rowland had to apply for the difference in the intensity of the horizontal component of the earth's magnetic force at the two galvanometers.

We have instead to determine exactly the ratio between two resistances of about 1 and 3000 B.A. units respectively, and to show that the heating of the wires by the current could never be such as to affect the value of this ratio appreciably. We proceed to describe the arrangement of the apparatus.

Fig. 1 gives a diagrammatic plan. A and B are the two coils, A being the primary. G is the galvanometer. K, L, M, N are four mercury cups.

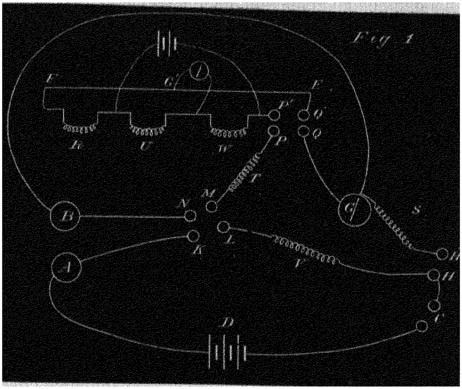


Fig. 1

P, Q, H, and H′ are also mercury cups. C is the commutator, and D the battery. Between H′ and G is a resistance, S, of about 3000 B.A. units, between H and L a second resistance, V, of about 1 unit.

Between P and M there is an adjustable resistance, T, the purpose of which will be described shortly.

The rest of the figure represents the ordinary Wheatstone's bridge arrangement for measuring the resistance of the secondary circuit. P′, Q′ are two mercury cups. E, F a divided wire. R a resistance of about 160 B.A. units (the total resistance of the secondary circuit). U and W two equal resistances of about 30 units, forming the other arms of the bridge.

Let us suppose P P′, Q Q′, and M N are connected by stout copper ┌┴┐-shaped pieces. Then our secondary circuit, broken between P and Q, forms the fourth arm of the Wheatstone's bridge, and by adjusting the variable resistance, T, the resistance of the secondary circuit can be made so nearly equal to R that the difference between them may be expressed in terms of the resistance of the bridge wire in the ordinary manner.

R then forms our standard resistance, and is the quantity actually measured in the experiments. R is the resistance of a coil of platinum-silver wire made at Professor STUART'S workshops, according to the pattern designed by Professor FLEMING, of Nottingham.

The wire was supplied by Messrs. ELLIOTT Brothers, and is that used by them in the manufacture of coils of about 100 units resistance. R, as has been stated, is about 160 units.

Now suppose that the connexions P P′, Q Q′ are broken, and that P Q is connected by means of one of the ┌┴┐ pieces. Break M N and connect L K, then the battery circuit is complete. Now connect M N again, the secondary circuit is complete, and the current running in the primary. Reverse the commutator C, an induction current is produced in the secondary circuit, and may be measured by the throw of the galvanometer needle.

To measure the primary current M N is broken and M L and H H′ are connected.

Between H and L the primary current is divided; the resistance in the direct circuit H V L being about 1 ohm, that in the circuit H S G Q P M L about 3072 ohms, so that about $\frac{1}{3073}$ of the whole current runs through the galvanometer; the permanent deflection of the galvanometer G is observed, and from this the value of the current is calculated.

We turn now to a detailed description of the apparatus used.

The coils were wound by Professor CHRYSTAL.

Two brass rings were carefully turned and a rectangular channel cut in the outer limb of each. A slit was cut in each ring to prevent currents in the frame (this was of course unnecessary for our experiments, but might render the coils more useful in many cases); the slit was closed with a piece of insulating material into which binding-screws connected with the wires of the coil are screwed.

The coils are lettered A and B.

In the experiments the coils are to be placed with their planes parallel and their centres on a line at right angles to their planes.

A cylindrical brass rod was carefully turned and fixed to a brass support so that it would stand in a vertical position.

Two tripod pieces of brass of the form shown in fig. 2 were constructed. The brass rod fits accurately through the aperture O in the centre of the tripod, and the plane of the arms O A, O B, O C is at right angles to the axis of the rod. The curved surfaces A A_1, B B_1, C C_1, are small portions of the same circular cylinder whose axis coincides with that of the rod, and the radius of this cylinder is the same as that of the carefully turned inner surface of the annulus on which the coils were wound.

This annulus thus would then fit on to A A_1, B B_1, C C_1 and rest on the flat surfaces A A_1 A_2 A_3, B B_1 B_2 B_3, C C_1 C_2, C_3, which are all parts of the same plane at right angles to the axis of the bar. If then we place one coil, A suppose, in this manner on the tripod, its plane is perpendicular to the axis of the bar, and its centre lies on that axis.

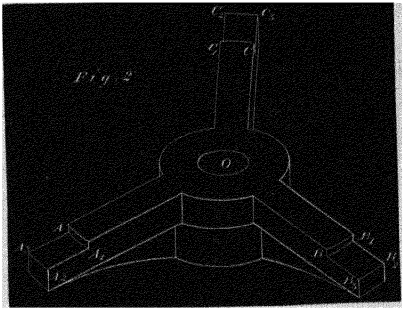


Fig. 2

To place the second coil with its plane parallel to the first we used three cylinders of brass, the ends of which had been carefully turned so as to be at right angles to their axes, while the lengths of the cylinders were as nearly as possible the same.

These three cylinders were placed vertically on the upper side of the coil A, one above each of the arms of the tripod, and the coil B rested on them ; its plane thus was as nearly as may be parallel to that of A.

To bring its centre into the axis a second tripod exactly similar to the first was placed in an inverted position over it, and the coil was moved about until the flange pieces fitted inside the inner surface of the annulus as before. In this manner the coils were adjusted to the required position.

The following account of the precautions used in the winding, and the methods employed to measure the constants of the coils, is quoted from a letter of Professor CHRYSTAL's, addressed to one of the authors of this paper (R. T. G.) :—

"The coils, stand, &c., were constructed after working drawings made by myself under the supervision of Professor MAXWELL. The immediate end in view in constructing the coils was the determination of the ohm, and this of course influenced the design of the stand. It was proposed ultimately, to use the coils as a standard instrument for producing a uniform magnetic field in which to determine galvanometer constants and the like.

"The coils were wound by myself and the then mechanic at the laboratory, Mr. FULCHER. The coils were mounted for this purpose by placing between the three armed supports"—the tripods mentioned above—"which were then braced together and mounted on an axle and stand. During the winding constant tests were taken for the insulation between the wire and the metal channel. This was the main difficulty, and wherever the slightest defect was discovered the wire was unwound for a little way and paraffin paper and paraffin used. It was found absolutely necessary, in order to secure good insulation, to cover the bottom of the channel with a ribbon of silk drawn through melted paraffin. The number of turns in each layer was separately counted and registered, and as a check a counter was attached to the axle and read at the end of the windings; the two records agreed in both cases. The resistance of the coils after winding was 84·9 and 82·7 B.A. units respectively. Four diameters were measured in every layer.

"The measurements of length were made by means of the cathetometer, and I find in my book a record of a comparison between it and the beam-compass scale, which had been tested* I believe; there are also measurements of the thicknesses of the walls of the channel and its width."

The numbers actually used in calculating the value of M were furnished by Professor CHRYSTAL from these measurements. The error between the cathetometer and beam-compass appeared in one or two measurements to amount to 1 in 2000, but in the majority of cases it was very much less, its mean value being, perhaps, 1 in 10,000, irrespective of sign. On the whole then we may, without sensible error, treat the cathetometer scale as accurate.

In another letter, Professor CHRYSTAL gives the following extract from his note-book of the direct results of the observations. He says: "You remember that each diameter is given as the mean of four.

"The first diameter is through the slit, the next 45° from it in the direction of the sun's motion, the letter on the coil being up, and so on.

"Here is the entry in my book for the fourth layer in coil A.

No. of layer.	No of turns	Cathetometer.	Cathetometer.	Difference	Mean.
4	26	781·40 781·30 781·27 780·60	280·14 279 95 279·58 279·52	501 26 501 35 501 69 501 08	501 35

"You will observe in the above extract that the two intermediate diameters are greater. This happens in most layers. At any rate the diameter perpendicular to that through the slit is in the great majority of cases the greatest, as might be expected."

Professor CHRYSTAL'S measurements then gave us as the mean of four observations in different positions the value of the external diameter of each winding, and also the total number of windings in each layer. In each coil there were 30 layers and in each layer about 26 windings. In coil A the total number of windings was 797 and in coil B it was 791.

Let the external diameter of the layers be d_1, d_2, d_3, &c., and let the number of turns in a layer be $26+n_1$, $26+n_2$, &c.

---

* This beam-compass has again been tested by Mr. DODDS during the present year and found correct. All our measurements of length are referred to it. (Nov., 1882.)

Let t be the thickness of the wire and silk used, and let A and a be the mean radii of the coils.

Then $2\mathrm{A}+t=\dfrac{26\{d_1+d_2+\ldots+\}+n_1d_1+n_2d_2+\ldots}{797}$

Now we know the diameter of the channel before the first layer was put on and also the diameter of the first layer; they were respectively

$$49{\cdot}565 \text{ and } 49{\cdot}728 \text{ centims.}$$

From this we find

$$t={\cdot}0815 \text{ centim.}$$

and finally

$$\mathrm{A}=25{\cdot}753 \text{ centims.}$$

The observations on coil B gave the same value for the thickness of the wire and covering, and we get for it

$$a=25{\cdot}766 \text{ centims.}$$

The method here adopted to determine the value of the mean radius allows for the fact that in winding one layer may sink somewhat into the one beneath.

Let the figure (fig. 3) represent a section of the coils by a vertical plane through the axis; let the coils be placed with their lettered sides down as in the figure. This we call position 1 throughout.

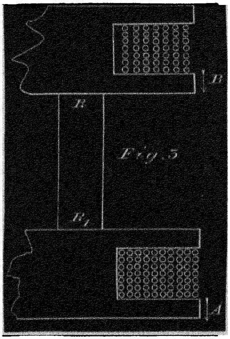


Fig. 3

In position 2 the lettered side of B was turned uppermost.

In position 3 the lettered side of A also was uppermost, while in position 4, A remained uppermost while B was again inverted.

Thus if $\downarrow$B mean that the lettered side of B was down, we have

Position 1 $\left\{\begin{matrix}\downarrow B\\ \downarrow A\end{matrix}\right.$ Position 2 $\left\{\begin{matrix}\uparrow B\\ \downarrow A\end{matrix}\right.$

Position 3 $\left\{\begin{matrix}\uparrow B\\ \uparrow A\end{matrix}\right.$ Position 4 $\left\{\begin{matrix}\downarrow B\\ \uparrow A\end{matrix}\right.$

In the figure R R_1 represents a section of one of the cylindrical rods used to separate the coils; its length and the thickness of the walls of the channel are all known. If, then, we assume that the median plane of each coil is midway between the walls of the channel, we can find the distance between the median planes; this we denote by b. But if the median plane in one coil, B for example, be not halfway between the walls but some small distance, δb suppose, nearer the lower wall, then in position 1 the value assumed for b is too large by δb, while in position 2 it is too small by δb, and by making observations in positions 1 and 2 and taking their mean we eliminate any error in the result which depends on the first power of δb. Similarly, if there be an error in the position of the median plane of the coil A of the same nature it is eliminated by inverting A.

The thicknesses of the walls of the channel as given by Professor CHRYSTAL are:—

	Coil A.	Coil B
	centim	centim.
Lettered side. . . .	·478	·446
Unlettered side . . .	·488	·465

Also, if $2h$, $2k$ be the radial and axial dimensions of the channel for coil A, $2h'$, $2k'$ for coil B, we have

$$h = \cdot 96 \qquad k = \cdot 95$$
$$h' = \cdot 95 \qquad k' = \cdot 95$$

Three series of brass cylinders were used to separate the coils, and the lengths of these were measured each by two observers. A pair of calipers graduated to read with a vernier to $\frac{1}{1000}$th part of an inch were found to be the most convenient instrument with which to make the measurements. The scale of the calipers was tested against the beam-compass without discovering any errors that could affect the result to as much as 1 in 10,000.

The following is the series of measurements for the longest rods used (Series C), made by R. T. G. and given as an example of the agreement of the observations.

Calling the rods α, β, γ, we found

α.	β.	γ.
inches.	inches.	inches.
9·389	9·388	9·390
9·388	9·389	9·391
9·388	9·389	9·390

It was noticed, however, that two of these rods had been slightly bruised at the end, thus producing a small lump at one point; when the rods were placed so that this lump came between the jaws of the calipers an increase of ·002 inch was observed in the length. In the calculations this greater length has been used as the true length of these two rods. In the shortest rods, series A, the difference in length, arising from a slight lump on the end of one of the rods, was somewhat greater, being about ·003 inch.

Another series of measurements made in an entirely different manner by placing first the rods and then the beam-compass beneath a pair of reading microscopes gave very closely concordant results; these measurements made the rods appear about ·001 inch longer than the measurements with the calipers. Some difference of this kind was to be expected from the difference which exists between the contact length and the sight length of a rod.

In our calculations we have taken the values given by the calipers.

Reducing them to centimetres we have for the lengths of the rods

	centims.
Series A	12·182
Series B	15·416
Series C	23·856

We have thus obtained all the dimensions requisite for the calculation of the mutual induction between the coils in the three series.

Let us call b the distance between the mean planes; b, of course, is slightly different in each of the four positions included in each series.

The calculations of M have been conducted as follows :—

If all the windings are supposed to be coincident with the mean windings, and M_0 be the mutual induction on this hypothesis

$$M_0 = 4\pi n n' \sqrt{Aa} \left\{ \left(c - \frac{2}{c} \right) F + \frac{2}{c} E \right\}$$

where

$$c = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + b^2}} \text{ and } n,\ n' \text{ are the number of windings ;}$$

and F, E are complete elliptic integrals to modulus c (Maxwell, vol. ii., § 701).

Appendix i., ch. xiv., to the second edition of Maxwell's 'Electricity' contains a table in which the logarithms of $M_0/4\pi\sqrt{Aa}$ are given for values of $\sin^{-1} c$ from 60° to 90°, proceeding by intervals of 6′.

The proper value of $\gamma = \sin^{-1} c$ is most easily obtained from the equations

$$r_1^2 = (A+a)^2 + b^2 \qquad r_2^2 = (A-a)^2 + b^2$$

$$\cos\gamma = r_2/r_1.$$

Thus when the coils are separated by the long rods

$$r_1^2=3365{\cdot}64 \qquad r_2^2=712{\cdot}36$$

whence

$$\gamma=62^\circ\ 36'\ 32''.$$

From the table

$$\log \mathrm{M}_0/4\pi\sqrt{\mathrm{A}a}=\bar{1}{\cdot}5718069$$

and this gives, taking A as 25·750, a as 25·760

$$\mathrm{M}_0={\cdot}761225\times 10^8.$$

When, however, the dimensions h, k of the section are too large in comparison with b to allow us to mass all the windings together, we may use a formula of approximation due to Mr. H. J. Purkiss (Appendix ii., ch. xiv., Maxwell, second edition).

Let the suffix $\pm h$ denote that for a is substituted $a\pm h$ in the corresponding functions, $\pm k$ that $b\pm k$ replaces b, and similarly for accented letters. Then

$$\mathrm{M}=\tfrac{1}{6}\{\mathrm{M}_h+\mathrm{M}_{-h}+\mathrm{M}_{h'}+\mathrm{M}_{-h'}+\mathrm{M}_k+\mathrm{M}_{-k}+\mathrm{M}_{k'}+\mathrm{M}_{-k'}-2\mathrm{M}_0\}.$$

To calculate these eight quantities in a methodical manner we notice that

$$\left.\begin{array}{l} r_{1h}^2=r_1^2+2(\mathrm{A}+a)h+h^2 \\ r_{2h}^2=r_2^2-2(\mathrm{A}+a)h+h^2 \\ \&\mathrm{c}. \end{array}\right\}$$

and that consequently we have to add in each case a correcting square, and add or subtract a correcting product. The corresponding γ is found, and the rest of the calculation effected just as for the mean windings.

When the coils are separated by the long rods the greatest and least values of γ are $\gamma_{-k}=63^\circ\ 26'\ 54''$, and $\gamma_k=61^\circ\ 46'\ 56''$, the final value result being

$$\mathrm{M}={\cdot}761921\times 10^8.$$

It thus differs by nearly 1 in 1000 from the uncorrected value.

The values of b differ slightly according to the four positions of the coils, and a slight correction has to be made in the values of A, a assumed above.

It therefore becomes of importance to determine the correction in M to be made for the addition of ·001 centim. to A, a, or b.

Since $\cos\gamma=r_2/r_1$, it follows that

$$\delta_a\gamma=\delta_\mathrm{A}\gamma={\cdot}001(\mathrm{A}+a)/r_1^2\tan\gamma \qquad \text{in circular measure.}$$

This gives for the long rods

$$\delta_a\gamma=1{\cdot}64''.$$

Again

$$\delta_b\gamma = \cdot 001(r_1^2 - r_2^2)/r_1^2 r_2 \tan\gamma$$

whence

$$\delta_b\gamma = -3\cdot 16''.$$

From the tables we find that when $\gamma = 62^\circ\ 36'$, the difference in the value of $\log M_0/4\pi\sqrt{a}A$ for an addition $1''$ to γ is 77×10^{-7}; that is, since $\delta M = \delta M_0$ very approximately,

$$\delta_a M/M = \frac{\delta a}{2a} + 77 \times 10^{-7} \times 1\cdot 64'' \times \log_e 10 = \cdot 000048$$

$$\delta_b M/M = -77 \times 10^{-7} \times 3\cdot 16'' \times \log_e 10 = -\cdot 000041$$

The true values of the radii are $A = 25\cdot 753$, $a' = 25\cdot 766$. The correction on this account to the value of M is therefore $9 \times \cdot 000036 \times 10^8 = \cdot 000324 \times 10^8$.

The corrections to be applied to b are:—In position 1, $-\cdot 008$; in position 2, $+\cdot 016$; in position 3, $\times\cdot 006$; in position 4, $-\cdot 013$.

Making similar calculations for the medium and short rods we can present the values of the mutual induction thus:

TABLE giving values of the mutual coefficient of induction between the coils.

Position of coils.	Experiments A. Short rods.	Experiments B. Medium rods.	Experiments C. Long rods.
1	$1\cdot 55636 \times 10^8$	$1\cdot 25797 \times 10^8$	$\cdot 762107 \times 10^8$
2	$1\ 55430 \times 10^8$	$1\cdot 25649 \times 10^8$	$\cdot 761321 \times 10^8$
3	$1\cdot 55539 \times 10^8$	$1\ 25727 \times 10^8$	$\cdot 761735 \times 10^8$
4	$1\cdot 55744 \times 10^8$	$1\ 25875 \times 10^8$	$\cdot 762521 \times 10^8$
Mean of the four . .	$1\cdot 55587 \times 10^8$	$1\ 25758 \times 10^8$	$\cdot 761921 \times 10^8$
Error in M produced by an error ·001 in a or A . .	$\cdot 000063 \times 10^8$	$\cdot 000052 \times 10^8$	$\cdot 000036 \times 10^8$
Error in M produced by an error ·001 in b	$-\cdot 000108 \times 10^8$	$-\cdot 000078 \times 10^8$	$-\cdot 000041 \times 10^8$

Thus the error produced by an error of ·001 centim. in b lies between ·005 and ·006 per cent., and the error in the measurement of b is certainly not more than ·001 inch or ·0025 centim.

Since the rods used to separate the coils were not exactly of the same length, the median planes cannot have been exactly parallel. The difference in the length of the rods is not as much as ·005 centim.

The radii of the coils are approximately 25 centims., and hence the angle between them is not as great as $\frac{\cdot 005}{37\cdot 5}$ or $\frac{1}{7500}$.

If we remember that when the coils are parallel the value of M is a maximum, so that the error due to the small angle between them depends on the square of the angle, it is clear that in our case this error is vanishingly small.

The galvanometer also was designed by Professor MAXWELL, and wound by Professor CHRYSTAL. It is also referred to in the article "Galvanometer," in the 'Encyclopædia Brittanica,' 9th edition. The description of it is taken from his account in the laboratory book, dated July, 1876.

There are two channels of rectangular section, and the following approximate dimensions :—

	inches
Depth of channel	$1\frac{1}{16}$
External diameter of bobbin	4
Breadth of channel	$0\frac{7}{8}$
Distance between channels	$0\frac{23}{32}$

Each channel contains 20 layers of thin copper wire and 16 layers of thick, making about 465 and 202 double turns respectively, so that there are 667 double turns in each channel, and about 2668 single turns on the galvanometer.

	inches.
Diameter of copper in thin wire	·014
Silk and all 82 thicknesses lie in	$1\frac{3}{8}$
Diameter of copper in thick wire	·029
Silk and all 34 thicknesses lie in	$1\frac{1}{6}$

The two thicknesses of wire were employed in order to fill the channels, and at the same time permit the resistance of the galvanometer to be reduced to the requisite amount. The ends of the wires are connected to binding screws on the bobbin marked A, B, &c., *a*, *b*, &c. A to *a* is one wire, B to *b* another. In our experiments the coils were connected up in series, the total resistance being about 60 ohms at a temperature of 13°·2 C.

The needle of the galvanometer was suspended from the WEBER suspension by three single cocoon fibres of 60 centims. in length.

The magnet was a small bar of hardened steel 1·5 centim. long, ·6 centim. broad, and ·12 centim. thick; its weight was ·708 grm. The magnet was attached by two small screws to a brass stirrup to which the mirror was fixed. A piece of brass wire 5·6 centims. long, with a screw thread cut on it, was fixed to this stirrup at right angles to the plane of the mirror, projecting equally on either side of the mirror. Two small brass cylinders could be screwed along this brass wire, and by means of them the moment of inertia and time of swing of the needle could be adjusted as required. The stirrup and mirror weighed 6·6 grms.

The galvanometer rested on a solid wooden base of about 18 centims. diameter, and this base was supported on three levelling screws.

A graduated circle is fixed to this stand, and the coils can be turned about a vertical axis, and their position read by means of a vernier. This was found useful in adjusting the coils parallel to the magnetic meridian. The galvanometer rested on a stone bracket built into the wall of the room. A scale placed approximately north and south at a distance of about 259 centims. from the magnet was reflected in the mirror and viewed through a telescope.

The scale rested on a solid wooden support on the floor of the room. The mirror, about 1·5 centim. square, was a specially good one, selected by a fortunate chance from among a number in the laboratory. The divisions of the scale were in millimetres, and after practice these could be subdivided by the eye with great accuracy to tenths. The scale itself was of paper; though this material is unsuitable for many purposes because of the changes produced by the weather in it, in our experiments these changes are of small consequence, for we require only the ratio of the throw produced by the induction current to the steady deflection produced by the permanent current; and the time which elapsed between the measurements was only a few minutes. Any shrinking or alteration of the scale will go on very approximately uniformly throughout its length and not alter the ratio of two lengths, which were never very unequal, as measured by the scale. After use the scale was carefully compared with the standard metre at the Cavendish Laboratory and the necessary correction applied to the readings.

The distance between the mirror and the scale only enters our result in the small correction necessary to reduce the scale readings so as to give the ratio of the sine of half the throw to the tangent of the deflection. It was unnecessary, therefore, to measure it with any great accuracy or to take steps to ensure its remaining the same from day to day; so long as it did not change during the half hour occupied by each experiment, all the conditions required by us were satisfied.

The resistance coils.

The standard coil R has been already referred to; the means adopted to measure its resistance will be described later.

Its value at a temperature of 14°·6 C. was found in May, 1881, to be 160·821 ohms.

The coil V used as a shunt to the galvanometer was made of thick German-silver wire. About 450 centims. of wire covered with silk were employed. The extremities of this were soldered to two stout copper rods with amalgamated ends, connexion with the rest of the apparatus being made by means of mercury cups; the ends of the rods were pressed down on to amalgamated pieces of copper at the bottom of the mercury cups.

The value of V was determined by repeated comparison at different temperatures with the B.A. unit known as "Flat coil" in Professor CHRYSTAL's report (Brit. Ass. Rep., 1876). The value of the Flat coil in mean B.A. units was taken from Professor FLEMING's recent comparison of the B.A. units at the laboratory. We have

Value of Flat coil at 15° C.	1·00003
Temperature of coefficient . . .	·00028 per 1° C
Value of V at 15° C.	1·0015
Temperature of coefficient . . .	·0003 per 1° C.

The coil itself is enclosed in a brass case and could be placed in a vessel of water. This was done during the experiment and the temperature noted by a thermometer graduated to fifths of a degree centigrade.

The coil S used with the galvanometer, when part of the battery current was sent through it, was a coil of platinum-silver wire of the ordinary form of about 3000 units resistance. It was immersed in the same vessel of water as V, and its temperature read by the same thermometer.

In our first series of observations the total value of the resistance M P Q G H, was observed and found to be 3072·38 ohms when the temperature of the coil S was 13·2.

The value of V at this temperature is 1·0011 ohms.

One extremity of the coil V dipped into the same mercury cup H as one extremity of the coil S, and the battery was also connected with this cup. The other extremity of V was connected by means of a piece of copper wire with L, the mercury cup in which the two portions of the battery current again united. This piece of copper wire was found to have a resistance of ·01556 ohm, so that the value of the resistance in the circuit H V L is 1·0167 ohms, at a temperature of 13°·2, and the currents in the two branches H V L and H G M L respectively, will be in the ratio of 3072·38 to 1·0167, so that if i be the battery current, that passing through the galvanometer will be $\frac{1\cdot0167}{3073\cdot39}i$.

T, as has been explained, was a variable resistance which could be adjusted so as to keep the difference between the resistance of the secondary circuit and the standard R sufficiently small to be measured in terms of part of the wire of the bridge E F.

During the experiments T had to be varied by somewhat over half a B.A. unit. Now T enters with S into the galvanometer circuit. The resistance, therefore, of this circuit was not quite the same during the observations, but varied by somewhat over ·25 ohm from its mean value, 3072·38 ohms.

The resistances W and U were two coils of about 30 ohms each wound on the same bobbin, and made of the same wire.

The galvanometer used with the WHEATSTONE'S bridge was one of about 150 ohms resistance, made by Professor STUART at the mechanical workshops, Cambridge.

Theory of the experiments.

Let $\overline{R}$ be the absolute resistance of the secondary circuit including the galvanometer, M the coefficient of mutual induction between the coils, and i the current in the primary circuit, then the total induced current produced by reversing the primary is $\frac{2Mi}{R}$.

Let β be the first throw of the galvanometer needle produced by this reversal, T the time of a complete vibration, λ the coefficient of damping, G the galvanometer constant, and H the horizontal intensity of the earth's magnetism.

Moreover let τ be the coefficient of torsion of the suspending fibre.

Then,

$$\frac{2Mi}{R}=\frac{H(1+\tau)}{G}\,\frac{T}{\pi}\left(1+\frac{\lambda}{2}\right)\sin\frac{\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, let a current i' be passed through the same galvanometer directly afterwards, and let θ be the permanent deflection produced.

Then

$$i'=\frac{H(1+\tau)}{G}\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$\overline{R}=\frac{2\pi M}{T\left(1+\frac{\lambda}{2}\right)}\cdot\frac{i}{i'}\cdot\frac{\tan\theta}{\sin\frac{\beta}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But

$$\frac{i}{i'}=\frac{S+V}{V}$$

Therefore

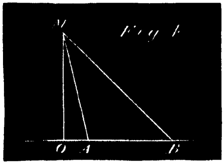
$$\overline{R}=\frac{2\pi M}{T\left(1+\frac{\lambda}{2}\right)}\cdot\frac{S+V}{V}\cdot\frac{\tan\theta}{\sin\frac{\beta}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

T is, of course, the observed time of oscillation.

The correction for the finite amplitude of the swing is too small to produce any error in the result. No correction for damping or torsion is required.

In the experiments the deflections on the scale were measured, not the angles of deflection. We require, then, to reduce the scale readings to angular measure.

Let p and q be the scale values of the throw and deflection, z the distance of the point of the scale vertically below the axis of the telescope from the point which appears to coincide with the cross wire when the needle is at rest. z being measured in the same direction as p and q, let a be the distance of the scale from the mirror.



Thus in the figure (fig. 4) let M be the centre of the mirror, O the point on the scale vertically below the axis of the telescope, A the point of the scale which appears to coincide with the cross wire when the needle is at rest, and B the apparent extremity of the throw.

Now the scale was carefully set so as to be at right angles to O M, and the distance O A was always small compared with O B.

Also

$$OA=z,\ AB=p,\ AMB=2\beta,\ OM=a.$$

Hence if we neglect squares of $\frac{z}{a}$ and fourth powers of $\frac{p}{a}$ and $\frac{q}{a}$

$$\text{angle } AMO=\frac{z}{a}$$

and

$$\tan\left(2\beta+\frac{z}{a}\right)=\frac{p+z}{a}$$

Therefore

$$\sin\frac{\beta}{2}=\frac{p}{4a}\left\{1-\frac{11}{32}\frac{p^2}{a^2}-\frac{pz}{a^2}\right\}$$

Also

$$\tan\left(2\theta+\frac{z}{a}\right)=\frac{q+z}{a}$$

hence

$$\tan\theta=\frac{q}{2a}\left\{1-\frac{q^2}{4a^2}-\frac{qz}{a^2}\right\}$$

and

$$\frac{\tan\theta}{\sin\frac{\beta}{2}}=\frac{2q}{p}\left\{1+\frac{11p^2-8q^2}{32a^2}+\frac{(p-q)z}{a^2}\right\}$$

Now the values of p, q, z, and a were such that $\frac{(p-q)z}{a^2}$ was about ·0001, we may therefore write with sufficient accuracy

$$\frac{\tan\theta}{\sin\frac{\beta}{2}}=\frac{2q}{p}\left(1+\frac{11p^2-8q^2}{3^2}\right)$$

p and q being the scale values of the throw and deflection.

To observe these quantities accurately the following adjustments are necessary.

The scale should be parallel to the mirror when at rest.

The coils of the galvanometer should be north and south so that their plane may be parallel to the magnetic axis of the needle.

The telescope should be placed so that its axis and the normal to the mirror, when at rest, may be in the same vertical plane. If this is the case the division of the scale which appears to coincide with the vertical cross-wire will be that just below the axis of the telescope, supposing, as in our case, the telescope is placed so as to look just over the edge of the scale.

In making these adjustments the scale and telescope were first fixed so that division 250—the middle of the scale—was vertically below the centre of the object-glass of the telescope; the galvanometer was placed in position and levelled, and the telescope adjusted to view the image of the scale in the mirror.

Then telescope and scale were both moved until the division of the scale which coincided with the cross-wire was close to 250. When this was the case the normal to the mirror and the axis of the telescope were nearly in the same vertical plane.

The scale was then turned in a horizontal plane until its two ends, equidistant from division 250, were also equidistant from the mirror, taking care at the same time that the image of division 250, the centre of the scale, remained close to the cross-wire of the telescope.

The scale was thus put at right angles to the normal to the mirror.

A long bar magnet resting on a pivot at its centre was then supported close to the scale, and it was found that the scale was very nearly parallel to the axis of the magnet, the error was certainly not more than 20′.

Thus the scale has been set very approximately north and south, and since the mirror is very nearly parallel to the scale, it is also nearly parallel to the axis of the galvanometer needle. Hence, if we set the coils parallel to the scale or mirror they will be very nearly north and south, and their plane will be approximately parallel to the axis of the needle.

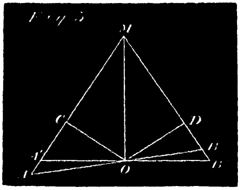
A piece of plate glass can be screwed on to the galvanometer in a position very nearly parallel to the coils. This was done, and the reflection of a lamp placed just below the telescope was observed, the galvanometer coils were turned until this reflected image was seen in the centre of the field of the telescope. Thus the galvanometer coils were placed very nearly north and south. The reading of the vernier attached to the galvanometer was noted, and by means of it the coils could readily be brought back to the same position, or placed at any required angle to the meridian.

The adjustments thus described were of course only approximate, but it is easy to show that the method of experiment eliminates any small outstanding error.

Let us suppose the coils are inclined at an angle α to the meridian. The effect of this is merely to change G into G $\cos\alpha$ in both the equations (1) and (2), and $\cos\alpha$ disappears from the resulting equation. It is better, however, that the field of force produced by the current in the galvanometer coil should be as nearly as possible uniform throughout the space through which the needle moves in the throw and deflection respectively. This condition is best satisfied if the needle when in equilibrium is parallel to the coils.

Let us now suppose that the scale is inclined at an angle γ to the plane of the mirror. Let p_1 and p_2 be the scale values of the throw to the right and left of the resting point.

Let A O B (fig. 5) be the scale, O M the normal to the mirror, draw A′ O B′ at right angles to O M.



Then A O A′$=\gamma$; let A and B be the observed extremities of the throw or deflection; θ its value in angular measure; let M A, M B, cut A′ B in A′ and B′, and let p be the true scale value of the deflection, then

$$\mathrm{A'O}=\mathrm{OB'}=p$$
$$\mathrm{AOM}=\mathrm{MOB}=2\theta$$

Draw O C, O D perpendicular to M A, M B respectively.

Then

$$\mathrm{OC}=\mathrm{OA'}\cos \mathrm{COA'}=\mathrm{OA}\cos \mathrm{COA}$$

Thus

$$p\cos 2\theta=p_1\cos(2\theta+\gamma)$$

Similarly

$$p\cos 2\theta=p_2(\cos 2\theta-\gamma)$$

Therefore

$$2p=(p_1+p_2)\cos\gamma-(p_1-p_2)\tan 2\theta\sin\gamma$$

Now p_1 was always very nearly equal to p_2, and 2θ and γ are both small.

Hence very approximately indeed we have

$$p=\frac{p_1+p_2}{2}\cos\gamma$$

Similarly

$$q=\frac{q_1+q_2}{2}\cos\gamma$$

Thus, by observing deflections right and left and taking the mean, we get a value for the ratio of p/q which is independent of an error in the azimuth of the scale, much greater than anything possible in the actual experiments.

If the magnet be not parallel to the mirror the angle turned through by the mirror is still that turned through by the magnet; the fact that the magnet and mirror were very nearly parallel afforded a ready means of setting the plane of the coils in the magnetic meridian.

The time of a complete vibration was measured in the usual way by noting the times of 8 or 10 transits of the resting point over the cross wire of the telescope, then waiting for the period occupied by some 10 or 12 oscillations and again observing the times of 8 or 10 transits. The value thus determined requires reducing to that for an infinitely small arc.

Now we know that if during the observation the arc of oscillation change from α_1 to α_2, and if T′ be the observed time of oscillation, then

$$\mathrm{T}=\mathrm{T}'\left\{1-\frac{1}{8}\left(\sin^2\frac{\alpha_1}{4}+\sin^2\frac{\alpha_2}{4}\right)\right\}$$

neglecting higher powers.

In the observations the value of α_1 was about 3°, that of α_2 about 1°·30′, the correction thus amounts to ·000025 and is quite inappreciable.

The value of λ was obtained by setting the needle vibrating, the secondary circuit being closed, and observing a series of resting points. If p_1, p_n be the amplitudes of the first and n^{th} vibration we have

$$\lambda=\frac{1}{n-1}\log_e\left(\frac{p_1}{p_n}\right)$$

Two independent observations of 17 vibrations gave

$$\frac{p_1}{p_{17}}=1{\cdot}2913 \text{ and } 1{\cdot}2909$$

Whence

$$\lambda={\cdot}0159$$

$\bar{\mathrm{R}}$ is the absolute resistance of the secondary circuit; this is very nearly but not quite equal to R, the resistance of our standard coil, and the difference between the two can be expressed in terms of the resistance of the wire of the B A bridge. This wire is 1 metre in length, and is divided into millimetres; let ρ be the resistance of 1 millim. The wire is graduated from E to F, (fig. 1); let G_1 be the position of the sliding contact piece when there is no current through the galvanometer, P P′, Q Q′, and M N being connected.

Let

$$\mathrm{EG}_1=x \text{ millim.} \qquad \mathrm{FG}_1=y \text{ millim.}$$

$\bar{\mathrm{R}}$ is the resistance of the circuit Q′ Q G B N M P P′.

Hence

$$\frac{\mathrm{R}+x\rho}{\mathrm{R}+y\rho}=\frac{\mathrm{W}}{\mathrm{U}}$$

Now interchange U and W, and let x' y' be the new values of x and y.

Thus

$$\frac{\overline{\mathrm{R}}+x'\rho}{\mathrm{R}+y'\rho}=\frac{\mathrm{U}}{\mathrm{W}}$$

But

$$x+y=x'+y'$$

Hence

$$\overline{\mathrm{R}}+x'\rho=\mathrm{R}+y\rho$$

$$\mathrm{R}=\overline{\mathrm{R}}+(x'-y)\rho$$

A number of experiments were made to determine ρ, and the value $\rho=\cdot000072$ B. A. unit was obtained.

Since ρ only comes in as a small correction, we may take one B. A. unit as 1 ohm.

Again, the value of R depends on the temperature, and our experiments required reducing to a constant temperature t_0; let t be the temperature of R at the time of experiment, R_0 the value of R at temperature t_0, and α the coefficient of increase of resistance per degree centigrade. Then we have

$$\mathrm{R}=\mathrm{R}_0\{1+\alpha(t-t_0)\}$$

Hence, finally our equation (4) becomes

$$\mathrm{R}_0\{1+\alpha(t-t_0)\}=\frac{2\pi\mathrm{M}}{\mathrm{T}\left(1+\frac{\lambda}{2}\right)}\cdot\times\frac{\mathrm{S}+\mathrm{V}}{\mathrm{V}}\times\frac{2(q_1+q_2)}{(p_1+p_2)}\left\{1+\frac{11p^2-8q^2}{32a^2}\right\}+(x'-y)\rho \quad . \quad . \quad (5)$$

The experiments were made in the following order :—

The time of swing was observed, the secondary circuit being closed as in the experiments.

The variable resistance in the secondary circuit was adjusted until the difference between $\overline{\mathrm{R}}$ and R could be measured in terms of the bridge-wire resistance, and the values of x y, x' y' determined. While this was being done a second observer read the temperatures of the coils R, S, and V, and the galvanometer G.

The connexion P P′, Q Q′ were broken, and P Q was joined. The resting point of the reflected image of the scale was observed when no current was passing through the galvanometer. This was done in the usual manner by observing five consecutive turning points.

The galvanometer needle was brought as nearly as possible to rest by the use of a damper. This consisted of a coil of wire placed near the needle, through which the current from a single LECLANCHÉ cell could be passed. By means of a second key a shunt could be introduced into this circuit so as to allow only a small fraction of the current from the battery to circulate in the coil. After a little practice the apparent oscillations of the galvanometer needle could easily be reduced to a few tenths of a millimetre of the scale.

When this had been accomplished, the battery current in the coil A was reversed and the first throw of the galvanometer needle observed. Suppose this was a throw to the right. By reversing the current at the right instant the needle could be brought very nearly to rest again, and the small swing that remained was easily destroyed by the damper. The connexions were then adjusted so that on again reversing the battery current a throw to the left was observed. A second throw to the left was observed and then a second throw to the right.

After this another reading was taken of the resting point when no current was passing.

The connexions were then altered so that a fraction of the direct current could be passed through the galvanometer, and the position of rest of the needle when the current was passing was observed. The needle was easily brought sufficiently nearly to rest in its new position by making the primary contact for a third of the time of swing, then breaking it for a second third, and finally making it again and leaving it made.

The resting point was determined while the needle was swinging from the observation of five consecutive turning points. The primary current was then reversed and a deflection in the other direction observed. After this a third reading of the resting point without any current was taken.

The connexions were again altered to observe the throws and four more were taken, one to the left, two to the right, and one to the left. A fourth observation of the resting point completed this part of the observations. The secondary circuit was then put into communication with the WHEATSTONE's bridge, and the difference between R and R measured, giving a second series of values of x y, x' and y', and finally the thermometers were all read again.

When we had become accustomed to the work a complete set of observations, excluding the time of swing, took about 25 minutes.

In the first series of measurements the time of swing was only observed twice each afternoon—at the beginning and end of the afternoon's work. In the second series it was taken generally at the beginning of the work and after every second or third set of observations, that is to say, at intervals of somewhat less than an hour and-a-half.

Throughout the experiments one observer (R. T. G.) read the galvanometer deflections, while the other (J. M. D. in the first series, E. B. S. in the second) made or broke the various connexions as required, and noted down the scale readings as they were read out by the observer at the telescope.

To obtain from the direct results of the observations the quantities required for substitution in formula (5), the following method was adopted:—

The means of the temperatures at the beginning and end of the observation were taken as the temperatures at the time of the experiment, and the means of the values of $x\ y$, $x'\ y'$.

The alteration in temperature during the experiment was only about ·1° or ·2° C. The changes in the values of $x\ y$, &c., during the observation were produced by the variation of the temperature of the secondary coil, which being copper has a high temperature coefficient. The alteration in the value of x was rarely as much as 300 bridge divisions, and since this change went on nearly uniformly during the experiment, the mean of the two values at the beginning and end will be very accurately the true value.

In reducing the scale readings to give the throws and deflections, we had to remember that owing to the variation in the direction of the earth's magnetic force, there was a continual change going on in the resting point of the needle.

In general, this change was only ·3 or ·4 millim. during the time occupied by a set of observations, sometimes it amounted to 1 millim. or rather over, and on one or two occasions during magnetic storms the changes were so violent and sudden that we had to cease work entirely.

For determining the value of the throw the following method enables us to eliminate the effect of this change in zero.

Take the mean of the two throw readings to the right, then the mean of the two to the left, and the mean of the resting point readings, then the differences between the throw readings and the resting point, or zero readings, will give the throws right and left respectively, corrected for change in resting point. The difference between the throw readings will give the value of p_1+p_2 directly; since, however, the throws right and left ought to be the same if the adjustments are correct—it forms a test of the accuracy of the measurements to calculate p_1 and p_2 separately.

For the permanent deflection, however, in which only one observation was made on either side of the zero, the same method is not applicable. The four observations we have to consider are: zero reading, deflection to right, deflection to left, zero reading.

Suppose the zero is moving from right to left, then, if we take the mean of the zeros and consider the differences between it and the deflection reading as the deflections right and left, in each case our deflection will be too great, while, if the zero be moving in the other direction, the deflection obtained will be too small.

To obviate the difficulty we assumed that the interval of time between each two consecutive observations was the same, and that the change in zero was uniform. We then obtained by interpolation the values of the zero readings at the moments of making the deflection observations. The differences between these and the deflection readings gave then the true deflections right and left, q_1 and q_2 respectively, the whole correction being a very small fraction of the measured deflection.

The second series of throws were then treated in the same manner as the first, and a second pair of values of p_1 and p_2 obtained. These generally differed somewhat from the first, for the electromotive force of the battery—a combination of DANIELL'S cells

—was not absolutely constant throughout an experiment. Now the observations of deflection refer to a moment of time about half way between the two throws, so that the mean of the two values of p_1+p_2 will give us the value of that quantity corresponding to the value of q_1+q_2 obtained for the deflections.

For the first series of observations the difference in the values of p_1 before and after the deflection reading was sometimes, but not often, as great as 1 millim. in a throw of about 215 millims.

A copy of the observations requisite for one experiment will perhaps render the above details more complete.

OBSERVATION No. II., June 8th, 1881, 11·45 a.m. Observers, R. T. G., J. M. D
Time of swing 23″·277.

Bridge reading value of x $\left\{\begin{array}{l}\text{U, W direct}\\\text{U, W interchanged}\end{array}\right\}$ 500
„ „ x' 410

Temperature.

R 13°·7 S 13°·7 Galv. 14°·5

Scale observations.

Zeros	235·5	
		227·5
	235	
		227·8
	235	
	Left.	Right.
Throws	7·0	
		455
		454·5
	7·8	
Zeros	225	
		238
	225·2	
		238
	225·2	

	Left.		Right.	
Deflections	87·2		364·8	
		94·5		377
	87·5		364·8	
		94·6		377
	87·8		365	

Zeros	237·8	
		224·2
	237·5	
		224·5
	237	

	Right.	Left.
Throws	454·2	
		6·0
		6·5
	454	

Zeros	234·5	
		227
	234·2	
		227
	234	

Temperatures.

R 13°·7 S 13°·8 Galv. 14° 8

Bridge reading value of x $\left\{\begin{array}{l}\text{Direct}\\ \text{Interchanged}\end{array}\right\}$ $\begin{array}{l}630\\ 540\end{array}$

„ „ x'

Battery: four ordinary DANIELL's in series.

From these observations we obtain the following value for the zero and deflection readings :—

Zeros	231·4	231·5	230·8	230·6

	Left.	Right.
Deflections	91·0	370·9

Thus the mean zero for the first throw is 231·45, and combining this with the observations of the throw we get

$$p_1 = 454{\cdot}75 - 231{\cdot}45 = 223{\cdot}3$$
$$p_2 = 231{\cdot}45 - 7{\cdot}4 = 224{\cdot}05$$

While for the second throw the zero is 230·7, and the values are

$$p_1 = 454{\cdot}1 - 230{\cdot}7 = 223{\cdot}4$$
$$p_2 = 230{\cdot}7 - 6{\cdot}25 = 224{\cdot}45$$

Thus the mean value of p_1 for the experiment is

223·35

and of p_2

224·25

These numbers require correcting for scale error.

The correction to p_1 is

+1·2

that to p_2 is

+1·0

so that

$$p_1 = 224{\cdot}55 \text{ millims.}$$
$$p_2 = 225{\cdot}25 \text{ millims.}$$

The difference between them being only ·7 millim. it is clear that the adjustments are all right.

Before taking the deflections the zero reading was 231·5, after taking them it had become 230·8. Thus interpolating, the zero reading at the moment of the deflection to the left was 231·3, and we find

$$q_2 = 231{\cdot}3 - 91{\cdot}0 = 140{\cdot}3$$

while at the moment of the deflection to the right the zero reading was 231·1, and, hence,

$$q_1 = 370{\cdot}9 - 231{\cdot}1 = 139{\cdot}8$$

Correcting for scale error we have

$$q_1 = 140{\cdot}6 \text{ millims.}$$
$$q_2 = 141{\cdot}1 \text{ millims.}$$

We have also

Mean value of temperatures—

R 13°·65   S 13°·7   Galv. 14°·65

Mean value x . . . ·565

„ x' . . . ·475

We proceed now to give in tabular form the results of our first series of experiments made in June, 1881.

The table will contain the values of $\frac{p_1+p_2}{2}$, $\frac{q_1+q_2}{2}$, the temperature of R, the mean values of x and x', and the time of swing. The corrections introduced by the variation in the temperature of S and the galvanometer we shall show, when we come to discuss the results, are so small that they may be neglected. We may, therefore, treat the results throughout as if the temperature of S were 13°·2, and that of the galvanometer 14°.

The battery used in all cases was four DANIELL'S cells.

The average difference between p_1 and p_2, irrespective of sign, was about ·32 millim., in one case only was it as great as ·9 millim.; it is hardly necessary, therefore, to give both p_1 and p_2 in the table.

The rods used to separate the two coils, primary and secondary, were the same, but, as has been explained, the coils were placed in each of four positions, numbered respectively I., II., III., and IV.

TABLE I.

Position.	Mean throw.	Mean deflection	T.	Temperature R.	x	x'.
I.	226·10	142·15	23″·277	15°·1	670	590
	226·25	142·40		15·2	583	500
	223·00	140·25		15·1	420	350
	222·70	139·70	23·250	15·2	540	470
	224·55	141·40	23·277	13·8	690	600
	224·90	140·85		13·7	440	355
II.	223·65	140·30		13·7	750	665
	222·80	140·20		13·9	505	420
	222·25	139·85		13·9	265	175
III.	218·00	137·05		14·2	770	670
	217·60	136·90		14·2	480	390
	217·50	136·70		14·3	325	240
IV.	216·05	135·85		14·3	778	680
	216·10	135·55		14·5	532	440
	215·85	135·50	23·274	14·5	400	310

The first four observations were made on June 7th, 1881, the last eleven on June 8th.

For the first four observations, therefore, we take the time of swing as 23″·264, for the last eleven as 23″·275.

The mean temperature of R during the experiments is about 14°·6; we take this then as the value of t_0. The temperature coefficient of R—a platinum-silver coil—may for the small range considered be taken as ·0003 per ohm per degree.

Thus the values of the constants in the formula (5) for R_0 are

$$t_0 = 14^\circ{\cdot}6 \qquad \alpha = {\cdot}0003$$
$$\tau = {\cdot}0007 \qquad \lambda = {\cdot}0159$$
$$a = 259 \text{ centims.}$$
$$\rho = {\cdot}000072 \text{ ohm.}$$
$$\frac{S+V}{V} = \frac{3073{\cdot}39}{1{\cdot}0167}$$

while giving p and q their mean values for the whole series of experiments, the term

$$\left\{1 + \frac{11p^2 - 8q^2}{32a^2}\right\} \text{ comes to } 1{\cdot}0018$$

As we have explained, the value of M is slightly different for each of the four positions.

The methods used to determine have been explained ; the values obtained were

$$M_1 = 1{\cdot}55636$$
$$M_2 = 1{\cdot}55430$$
$$M_3 = 1{\cdot}55539$$
$$M_4 = 1{\cdot}55744$$

where the suffixes refer to the positions.

Table II. gives the results of the calculations. As the results stand there, the negative errors are fewer in number than the positive, the two greatest errors being negative.

The greatest of these is ·453, which is about 1 part in 350.

TABLE II.

Position.	R_0 in ohms	Error from mean of set	Mean value of R_0 for each position	Error from mean of whole.	Percentage error.
I.	158·82	147	158·673	·047	029
	158·98	·307			
	158·84	·167			
	158·48	– 193			
	158·70	037			
	158·22	–·453			
II.	158·27	–·297	158·567	– 059	–·039
	158·68	·103			
	158·76	193			
III.	158·69	·093	158·597	– 029	– 018
	158·77	·173			
	158·33	–·267			
IV	158·87	·241	158·629	·003	+·002
	158·51	– 119			
	158·51	–·119			

Mean value	**158·626 ohms.**
Mean error of mean of each from mean of whole . .	**·037**
Mean percentage error	**·023**

We shall retain the whole series of observations and take as the resistance of our standard coil R at a temperature of 14°·6 C., the value

$$158{\cdot}626\ \frac{\text{earth quadrants}}{\text{second}}$$

It remains now to explain the method used to determine the values of the resistances.

For this purpose the coils in a post office resistance box, made by Messrs. ELLIOTT Brothers, were compared with the standards at the Cavendish Laboratory. The 1 unit coil of the box was compared with the coil known as Flat in Professor CHRYSTAL's report, then the 1-unit + Flat were balanced against the 2-unit coil of the box, then this 2-unit against the second 2-unit, which we will denote by 2′, then 1+2+2′ against the 5-unit coil, and so on.

In this manner all the coils between 1 and 2000 B.A. units were compared.

The British Association wire bridge was used in making the comparison.

In the ordinary use of this (CAREY-FOSTER's method) the two coils to be compared are connected to the ends of the bridge wire and a measurement taken, the coils are then interchanged and another observation is taken, and from these two the difference between the coils is expressed directly as the resistance of a portion of the bridge wire. We, however, could not apply this method, for, calling P and Q the coils to be compared, since P and Q are coils in the same box, one end of P is always in electrical connexion with one end of Q. The following arrangement therefore was adopted :—

Two coils of known resistance were connected one to each end of the bridge wire, while P and Q formed the other arms of the bridge. The coils actually used were those marked F and G in CHRYSTAL's report.

The sliding contact was adjusted till no current passed through the galvanometer, and its position noted.

Let $\alpha+x$ be the resistance of the portion of the wire connected with F, $\alpha-x$ of that connected with G, so that 2α is the whole resistance of the bridge wire. Let $1+\delta F$, $1+\delta G$ be the resistances of F and G at the temperature of the observation.

δF and δG are very small.

At 14°

$$\delta F=-{\cdot}00084 \text{ ohm}$$
$$\delta G=-{\cdot}00112 \quad ,,$$

Then we have

$$\frac{P}{Q}=\frac{1+\delta F+\alpha+x}{1+\delta G+\alpha-x}$$

Interchange F and G and let x' be the new value of x

$$\frac{P}{Q}=\frac{1+\delta G+\alpha+x'}{1+\delta F+\alpha-x'}$$

Whence $x+x'=\delta G-\delta F$.

Thus x and x' are exceedingly small, and if we neglect squares and higher powers of δF, δG, x, and x', we obtain

$$(1+\alpha)\frac{P-Q}{Q}=x-x'$$

Now $\alpha=\cdot036$ mean B.A. unit.

If then we know Q, the value of P can be found from the observations of x and x'.

Two series of observations were taken, one by R. T. G. the other by J. M. D., each observation in each series being the mean of 2 or 3.

The extreme difference between the two series was in no case more than 1 in 3000.

The box remained in the room for some time before taking the observations, and the temperature was supposed to be that of the room as indicated by a thermometer laid on the box. A small correction was made for the resistance of the copper rods which connected the box to the bridge and the plugs in the box. This was determined by one observer (R. T. G.) only, so that any error in it will affect both measurements equally. We shall show, however, shortly that it cannot affect the value of the B.A. unit as determined from our measurements. Having thus obtained the values of the resistances in the box in terms of the B.A. standard units, the value of R_0 in these units was determined by the ordinary method. We found thus the mean of several closely concordant measurements

$$R_0=160\cdot821 \text{ B.A. units}$$

the temperature being 14°·6 C.

The resistance S of our secondary circuit and galvanometer was determined in terms of the coils in the same box, and we found

$$S=3072\cdot38 \text{ B.A. units}$$

at a temperature of 13°·2 C.

Owing to the difficulty of determining the value of the resistance of the plugs in the box and the copper connecting pieces, either of these results may possibly have an error of about 1 in 1500. Now the value of S enters into the value of R_0 in absolute units and affects it in the same way.

To determine the value of the B.A. unit we require the ratio

$$\frac{R_0\,\dfrac{\text{earth quadrant}}{\text{second}}}{R_0 \text{ in B.A. units}}$$

Its value is

$$\frac{158\cdot626}{160\cdot821}\text{ ohms.}$$

Whence we get

$$1 \text{ B.A. unit}=\cdot98635 \text{ ohm}$$

But the error we have been considering affects in exactly the same manner the numerator and denominator of this ratio. If in consequence of it one is too great so also is the other in the same proportion. Thus the accuracy of our result is not impaired by the uncertainty of this correction. In fact, although to determine the resistance of our coil R in absolute measure we require to determine accurately a ratio of 1 to 3000, and this determination has generally been held to be one of the main objections to our method, yet to determine the value of the B.A. unit we have in addition to compare a ratio of 160 to 1. Thus, in fact, to determine the value of the B.A. unit the ratio to be compared is 160 to 3000, or about 1 to 19, and this is a much easier experiment to make.

Our 160 and 3000 have both been expressed in terms of the resistances of the box, and even though there may be some considerable error in the actual values of these resistances, the error in the ratio of any two of them is a quantity very small indeed.

In conclusion, we would refer to another objection which has been made to the method. Nearly the whole of the battery current is allowed to flow through the coil V, whose resistance is about 1 ohm; the effect of this must be to heat V and alter its resistance, thus producing error. We shall show that the error in our case is vanishingly small.

The electromotive force of the battery was at most about 5 volts, and the total resistance of the primary circuit was about 80 B.A. units. The coil V was of German-silver wire, about 450 centims. being used to make it; the wire thus was very thick, its radius being ·06 centim. The wire, silk covered, was loosely wound in a coil and enclosed in a brass case, which was immersed in water.

From these data we find that the amount of heat developed per minute in the coil will be ·055 unit.

If we suppose all this heat to be retained, the rise of temperature will be ·015° C. per 1′, and the increase of resistance ·0000045 B A. unit, and this will be too small tc affect our results. As a matter of fact, it is clearly impossible for all the heat to remain in the coil, and the correction is, *à fortiori*, too small to be considered.

During the second series of experiments a hole was bored in the brass case of the coil and a thermometer inserted. The thermometer agreed throughout in its readings with that in the water bath in which the coil was immersed.

Thus we conclude as the final result of this series of experiments that the value of the B.A. unit is ·98635 ohm.

The agreement between the individual experiments of the series is remarkable. They are, however, open to the objection that the conditions under which they were taken remained unaltered in some essential particulars. Thus the rods used to separate the primary and secondary coils were the same throughout, while the battery was also the same. It was decided, therefore, to make a second series of observations in which these quantities were varied. This was done during November and December, 1881. Mr. DODDS had left Cambridge, and his place was taken by Mr. E. B. SARGANT, of Trinity College.

PART II.

Profiting by our past experience, the arrangement of the apparatus was modified slightly.

In Part I., as has been explained, one electrode of the coil V dips into the mercury cup H, while the other is connected by means of a piece of copper wire with the cup L.

In Part II. the mercury cups and resistance coils were so placed that the second electrode of V dipped directly into the cup L; the piece of copper wire between the two, therefore, was dispensed with.

Again, in fig. 1 it will be seen that between P and M in the secondary circuit there is a variable resistance T, used to adjust the resistance of the circuit until it balanced R.

This variable resistance formed a part of one of the two paths open to the current when measuring the deflection, and, as we have said, renders the exact proportion into which the current was divided in each experiment uncertain to the amount of about 1 in 6000.

In fitting up the apparatus for Part II., T was placed between B and N. As before, the resistance of the secondary circuit could be adjusted, but that of the primary remained unaffected by alterations of T, which in this part formed no portion of it.

Three sets of rods were used to separate the primary and secondary coils, we shall call them A, B, and C, respectively. The rods A were those used in Part I.

With the rods A three different electromotive forces were used; the batteries employed being respectively four ordinary DANIELL'S, two ordinary DANIELL'S, and five THOMSON'S tray DANIELL'S.

In position B we had five THOMSON'S-DANIELL'S, and in position C five THOMSON'S-DANIELL'S and six THOMSON'S-DANIELL'S.

As before, the coils were placed in positions I., II., III., and IV., but the order of taking the observations was somewhat varied. In Part I. three observations were taken in each position without altering the coils; in Part II., however, after taking one observation in position I., one of the coils was reversed so as to bring them into position II., and an observation made; the other was then reversed, and so on, and after the four measurements had been taken the whole series was repeated. This method necessitated rather more handling of the coils than the other; it had, however, the advantage that each set of four observations was taken under more nearly similar conditions, while, in consequence of the more frequent setting of the coils, the error due to any one chance bad setting was reduced. The time of swing was observed more frequently, being taken twice and generally three times for each set of four. The times corresponding to the mean throw and deflection are given in the table, being obtained by interpolation from those actually observed.

Table III. gives the same observations for Part II. as are given in Table I. for Part I.

With reference to the table we must notice that the experiments were not always made in the order I., II., III., IV., and therefore the times of swing in the column headed T are not in order of magnitude. Thus, in the second set, the real order of the experiments was II., I., III., IV., and in this order the values of T increase uniformly by 0″·005. In the experiments with two DANIELL'S, experiment IV. was made on November 16th, experiment II. on November 18th, and experiments III. and I. on November 21st.

TABLE III.—Series A.

Battery.	Position.	Mean double throw.	Mean double deflection.	T.	Temperature R	x.	x'.
4 DANIELL'S .	I	359·1	223·35	23″·384	12°·4	495	400
	II	360·4	224·5	23·384	12·7	640	555
	III	362·75	225·2	23·386	12·8	630	540
	IV.	361·9	225·3	23·388	13·1	615	525
4 DANIELL'S	I	364·2	227·35	23·415	13·4	350	270
	II	364·55	227·7	23·410	13·4	300	215
	III.	359·4	223·8	23·420	13·3	615	530
	IV.	352·45	220·2	23·425	13·0	565	455
5 THOMSON'S .	I	451·2	280·8	23·373	11·9	510	420
	II	453·2	281·9	23·367	11·8	420	330
	III.	452·2	280·4	23·380	12·2	525	435
	IV.	450·9	280·8	23·378	12·1	515	430
2 DANIELL'S .	I.	190·55	119·1	23·386	12·4	575	490
	II	191·8	119·8	23·405	13·4	415	335
	III.	190·75	118·8	23·382	12·3	805	715
	IV.	191·3	119·6	23·391	13·4	570	490

TABLE III.—Series B.

Battery.	Position.	Mean double throw.	Mean double deflection.	T.	Temperature R.	x.	x'.
5 THOMSON'S .	I.	374·5	287·95	23″·344	11°·6	630	545
	II.	373·25	286·5	23·355	11·9	895	800
	III.	375·1	288·3	23·322	10·8	415	335
	IV.	374·15	286·9	23·333	11·2	780	700
5 THOMSON'S .	I.	370·9	285·6	23·406	12·6	575	485
	II.	372·0	286·3	23·405	12·5	465	385
	III.	369·15	285·15	23·407	13·0	480	380
	IV.	370·5	286·2	23·406	12·8	505	415

TABLE III.—Series C.

Battery.	Position.	Mean double throw.	Mean double deflection.	T.	Temperature R.	x.	x'.
5 THOMSON'S	I.	227·4	289·8	23″·379	12·4	560	420
	II	229·6	292·4	23·382	12·5	655	575
	III	228·0	290·7	23·384	12·8	705	620
	IV.	228·2	290·5	23·381	13·0	690	605
6 THOMSON'S	I	252·9	321·9	23·389	13·6	555	485
	II.	253·1	323·4	23·391	13·3	545	475
	III	254·7	323·9	23·393	13·2	445	340
	IV.	250·2	318·9	23·395	12·9	425	335
6 THOMSON'S .	I.	243·25	310·95	23·381	13·4	950	860
	II	242·8	309·8	23·374	12·7	680	590
	III.	240·75	307·1	23·392	14·0	915	830
	IV.	240·65	306·85	23·384	13·8	280	190

These direct experimental results require to be substituted in our formula in order that we may obtain the values of R_0. The coils were placed in the four positions I. to IV. in order to eliminate any small unknown error in the position of the mean plane.

Now our first series of experiments have been sufficient to show that this error must be exceedingly small, and the result obtained by taking the mean of the four will certainly eliminate it.

Instead, therefore, of giving the value of R_0 for each position, we shall only calculate the mean value for each set of experiments, using, of course, as our value of M, the mean of the values M_1, M_2, M_3, M_4.

The correction for damping is the same throughout the whole series of observations, the value of $1+\frac{11p^2-8q^2}{32a^2}$, of course, differs for the different currents used; it therefore is included in the table.

The value of M for each of the three series A, B, C, is also given in the table.

The value of λ found from a large series of closely concordant measurements was

$$\lambda = ·01368$$

In these experiments the temperature was somewhat lower, generally, than in Part I., so that the mean value of the temperature of R was about 12° and $t_0 = 12°$.

The distance between the mirror and the scale was different, and we found

$$a = 218 \text{ centims.}$$

also

$$\alpha = ·0003$$

$$\rho = ·000072 \text{ B.A. unit}$$

α being the coefficient of increase of resistance of R per B.A. unit per degree, ρ the resistance of 1 millim. of the bridge.

The values of the resistances S and V were different from those in Part I., for the piece of copper wire which was included in the value of V in Part I. had been removed while the variable resistance T was no longer in circuit with S.

The values employed in the calculations are

$$V = 1{\cdot}00096 \text{ units.}$$
$$S = 3059{\cdot}89 \quad ,,$$

so that

$$\frac{V+S}{V} = \frac{3060\,891}{1\,00096}$$

The temperature of V and the 3000 ohms, which was in the same water bath with it, being 12°, that of the galvanometer 13°·5; this latter temperature being the mean of the temperatures of the galvanometer during the observations.

As before, we shall describe later the methods used to determine these values.

Table IV. gives the results of the calculations.

The mean value of R_0 deduced from it is

$$158{\cdot}386 \frac{\text{earth quadrant}}{\text{second}}$$

As we shall see afterwards, the value of R_0 in B.A. units is 160·520.

Thus

$$1 \text{ B.A. unit} = \frac{158\,386}{160{\cdot}520} \text{ ohms}$$
$$= {\cdot}986706 \text{ ohm.}$$

TABLE IV.

Series.	Battery.	M.	$\frac{11p^2-8q^2}{82a^2}$	R_0 in ohms.	Error in R_0 from mean of series	Mean value of R_0 for series.	Error from mean.	Percentage error.
A	4 D.	1·55587 × 10^8	·00167	158 233	−·158	158 391	·005	·003
	4 D.		·00167	158·472	·081			
	5 TH.		·00263	158·296	−·095			
	2 D.		·00045	158 564	·173			
B	5 TH.	1·25758 × 10^8	·00136	158·171	−·099	158·270	−·116	− 072
	5 TH.		·00136	158·368	+·098			
C	5 TH.	·762092 × 10^8	−·00019	158·397	−·061	158·458	·072	·045
	6 TH.		−·00015	158·301	−·157			
	6 TH.		−·00015	158·676	·218			

Mean of whole series 158·386 ohms.
Mean error for each series ·064.
Mean percentage error ·040.

Taking each series separately, the values we obtain are

Series A ·98673.
„ B ·98598.
„ C ·98716.

The greatest difference is ·00118, or about ·12 per cent.

Again, if we refer to the Table IV., we cannot find any clear indication of an error depending on the length of the rods used to separate the coils. The values of R_0 in series B are, it is true, somewhat small; there are, however, two values in series A and one in series C which are smaller than one of those in series B.

Neither do we find any connexions between the differences and the electromotive force used. Arranging the values in order of electromotive force, we have

2 Daniell's	158·564	1 experiment.
4 „	158·352	Mean of 2.
5 Thomson's	158·209	„ 4.
6 „	158·488	„ 2.

We must notice, however, that only one experiment was made with the smallest electromotive force.

It will be instructive to arrange the results in order of magnitude, noting the series, the battery, and the date of each experiment.

We have

Date.	R_0.	Series.	Battery.	Error	Percentage error
Dec. 2	158 171	B	5 Thomson's	− 215	− 135
Nov. 16	158 233	A	4 Daniell's.	− 153	− 095
„ 21	158 296	A	4 „	− 090	− 056
„ 25	158 301	C	6 Thomson's	− 085	− 053
Dec. 2	158 368	B	5 „	− 018	− 011
Nov. 23	158 397	C	5 „	011	·007
„ 18	158 472	A	4 Daniell's.	086	053
„ 16, 18, and 21	158·564	A	2 „	178	111
„ 28	158 676	C	6 Thomson's	·290	·181

An inspection is sufficient to show that there is no definite order in any column but the second with the fifth and sixth, which are consequences of it. The greatest difference between any two experiments is ·505, and this in 158 is rather less than 1 in 300, or about ·32 per cent.

Of the actual errors of each experiment from the mean, five are negative and four are positive; the mean error itself is only ·125; the mean percentage error is ·078.

The number of experiments made is too small for the calculation of the probable error to have any value, but the distribution of errors round the mean is satisfactory, and the mean percentage error is very small, if we consider the complicated nature of the observations and the variation in the important conditions.

We turn now to the measurement of the resistances of the coil used in Part II. and their comparison with the B.A. standards.

It will be remembered that in the comparison between the coil Flat of the B.A. units and the 1 ohm of the box used in Part I., the correction to be made for the copper pieces connecting the box to the wire bridge and the resistance of the plugs was thought to introduce some error, which, however, it was shown would not affect seriously the value of the B.A. unit. To reduce this error the 1-unit coil of the box was not compared directly with the B.A. standard. Lord Rayleigh had had wound two 5-unit coils and one 10-unit, which had been carefully compared by him with the B.A. units. The comparison was repeated by one of us (R. T. G.), and the differences between the two results were found to be so small that we could use either value with all the accuracy required.

Thus the values at 12° were

5 units . . .	4·99392	Lord Rayleigh.
	4·99376	R. T. G.
10 units . . .	9·98360	Lord Rayleigh.
	9·98393	R. T. G.

The second 5-unit coil was only measured by Lord Rayleigh and Professor Fleming; it belonged to the latter, and had been taken away by him before our comparison was made.

A third 5-unit, denoted afterwards by 5′, however, had been constructed for the laboratory to replace it, and its value was found to be 5·00890 at 12° C., while Professor Fleming's coil had a resistance of 5·02444. Lord Rayleigh's value of the 10 units was found by comparison with the 5 units + Fleming's 5 units in series; our value was obtained by comparison with $5+5'$. The close agreement between the two results is sufficient test of the accuracy of the comparisons.

In determining the values of the resistances of the boxes, we started from these 5 and 10-unit coils. Two boxes were used—one by Messrs. Elliott Brothers, No. 229, the other by Warden and Muirhead, No. 202. The 10-unit coils in each of these boxes were compared with our 10-unit standard, using the modification of Carey-Foster's method already employed to compare R and the resistance of the secondary circuit. Then 20 units in the box, made up in three different ways (viz.: by taking out (*a*) plug 20; (*b*) plugs 10 and 10′; (*c*) plugs 1, 2, 2, 5, and 10), was compared with the 10-unit and two 5-unit standards in series.

A large number of determinations were made both by R. T. G. and E. B. S. on different occasions. The various values obtained for the Elliott box, reduced to a temperature of 12°, are given below; the temperature is that recorded by a therm meter laid on, or on some occasions inside, the box. Each number is the mean three or four measurements taken at the same time.

Plug out.	Value.	Observer.
10	9·9903	R. T. G.
	9·9902	R. T G.
	9·9890	E. B. S.
10′	9·9887	R. T. G.
	9·9907	R. T. G.
	9·9908	E. B. S
1+2+2+5	9·9870	E. B. S.
20	19·9772	R. T G.
	19·9775	E. B. S.
10+10′	19·9769	R. T. G.
1–10	19·9756	R. T. G

For the WARDEN and MUIRHEAD box the differences between two sets of expe ments were quantities of the same order as here.

The two boxes were then placed at opposite ends of the bridge-wire, the other tv arms of the WHEATSTONE'S bridge being the pair of 30-ohm coils used in the previo part of the experiments.

The 50-unit plug was taken out of one box, and out of the other all the plugs fro 1–20. The difference between these two nominal 50 units was thus obtained. Th plugs 1–20 were taken out of the first box, and the 50 units out of the second, a another difference obtained. In this manner the values of the coils 50, 100, 100′, a 200 were obtained.

After this the differences between the two boxes became too large to be measur in terms of the resistance of the bridge-wire, and recourse was had to the meth employed in Part I., by which one coil in a box was compared with a combination coils in the same box.

Two coils, each of about 5 ohms, were connected with the ends of the bridge-wir while the coils to be compared, P and Q, formed the other arms of the bridge.

Let $5+\alpha$, $5+\beta$ be the resistances of the two 5-ohm coils, and let x, y, x', y' ha the usual meanings.

Then

$$\frac{\mathrm{P}}{\mathrm{Q}}=\frac{5+\alpha+x}{5+\beta+y}$$

Intercharge the 5-ohm coils

$$\frac{\mathrm{P}}{\mathrm{Q}}=\frac{5+\beta+x'}{5+\alpha+y'}$$

Hence

$$\frac{P}{Q}=1+\frac{\alpha-\beta+x-y}{5}$$
$$=1-\frac{\alpha-\beta-x'+y'}{5}$$

neglecting $\left(\frac{x}{5}\right)^2$ and such terms.

Now the values of the 5 units being known, we know α and β, and we find at the temperature of the observation

$$\frac{\alpha-\beta}{5}=\cdot 0061$$

Hence

$$P-Q=Q\left\{\cdot 0061+\frac{x-y}{5}\right\}$$

and

$$P-Q=Q\left\{\frac{x'-y'}{5}-\cdot 0061\right\}$$

The values of x, y, &c., actually obtained were such as fully to justify the neglect of $\left(\frac{x}{5}\right)^2$ and such terms. By this means the values of the 500, 1000, 1000′, and 2000-unit coils in the ELLIOTT box were determined.

Having thus determined the values of the resistances of the box, they were used to determine that of R in the following manner.

R was connected with one end of the wire of the bridge, and the two boxes in multiple arc with the other, two 30-unit coils forming as before the third and fourth arms.

170 units were then taken out of the WARDEN box, and the ELLIOTT box adjusted until the difference between R and the total resistance of the compound circuit formed by the two boxes could be measured in terms of the bridge-wire. This was the case when 2920 units were out of the box. The actual value of these resistances at 12° is known from our table of resistances of the box. Making the correction for temperature and for the difference between R and the multiple arc resistance, we find

$$R_0=160\cdot 602 \text{ B.A. units.}$$

A second determination, in which 180 units were out of one box and 1490 in the other, gave

$$R_0=160\cdot 570 \text{ B.A. units.}$$

Each of these results is the mean of several experiments.

We may take, therefore, as the value of R_0 the mean

$$R_0=160\cdot 586 \text{ B.A. units.}$$

The value of S, the resistance of the circuit through which the fractional part of the current ran when the deflection of the galvanometer was observed, was determined by direct comparison with the ELLIOTT box.

The values

3060·71 B.A. units

and

3060·90 B.A. units

were obtained as the means of several determinations on two different occasions.

The mean is

$$S=3060{\cdot}80$$

The value of V was determined by comparison with the B.A. coil Flat, using the form of bridge devised by Professor FLEMING.

We found on four separate occasions the values

1·00096
1·00096
1·00095
1·00098

The temperature throughout at which these values are correct is 12°.

These methods involve the use of the resistance boxes, and, of course, there must be some uncertainty in the temperature of the coils in the interior of the box. Another series of determinations, therefore, was made of the values of R and S, in which the boxes were only used to determine the values of certain small corrections.

There is at the laboratory a coil of about 24 B.A. units—the standard used by Lord RAYLEIGH in his experiments on the value of the B.A. unit. This he had compared with the B.A. units and found that at 12° its value was 23·92820. He suggested that we should compare our standard of about 160 units directly with it, and this was done by one of us (R. T. G.).

I first determined *ab initio* the value of the 24 ohms in terms of the B.A. units. I found by a method to be afterwards described the three values

23·92807
23·92810

and

23·92850

The mean is

$$23{\cdot}9282\dot{2}$$

which is practically identical with that found some months previously by Lord RAYLEIGH.

Now the ratio of 24 to 160 is between 1 : 6 and 1 : 7.

I therefore arranged a WHEATSTONE's bridge in the following manner:—

Two arms were the two standards I wished to compare; the coil R of about 160 units and Lord RAYLEIGH'S standard, which we will call X, of about 24.

The third arm was two single unit coils and a 5-unit arranged in series, and the fourth arm a single unit, the coil Flat.

The two single units were the B.A. units C and G of Professor CHRYSTAL'S report, their values and that of the 5-unit are accurately known in terms of the mean B.A. unit.

In all cases the electrodes of the coils were well amalgamated and rested securely on the copper discs at the bottoms of mercury cups.

The electrodes of the ELLIOTT box were connected with the two ends of the 7-unit arm, so that the box formed a shunt, and by altering the plugs in the box the effective resistance of the arm could be finely adjusted, and the ratio of the resistance of the Flat coil to that of this arm made equal to that of R to X.

Thus I found that with 164 units out of the box there was a deflection of the galvanometer of $+33{\cdot}2$ scale divisions, while with 163 out the deflection was $-5{\cdot}6$.

Thus the true value of the shunt is 163·143. Correcting this to the proper temperature we find that the effective resistance of this third arm is 6·70438 units, that of the Flat coil at the same temperature being ·99944 unit.

Now the temperature of R at the time of observation was 13°·3, that of X being 13°·4.

Hence

$$\frac{R_{13\cdot 3}}{X_{13\cdot 4}}=\frac{6\cdot 70438}{\cdot 99944}$$

Whence substituting the value of X and reducing to the standard temperature 12°

$$R_0=160\cdot 523 \text{ B.A. units.}$$

A second experiment was made at a different temperature, and instead of noting the deflections of the galvanometer produced by altering the box by 1 unit a second shunt was introduced and varied until the deflection was zero: the value of this shunt was 30,000 units.

From this experiment I found

$$R_0=160\cdot 518 \text{ B.A. units.}$$

We take as the true value of R_0 deduced from these two experiments

$$R_0=160\cdot 520 \text{ B.A. units.}$$

We have now to compare directly the values of S_0 and R_0. S_0 is about 3060 units, so that the ratio of S_0 to R_0 is between 19 to 1 and 20 to 1.

Four sets of coils were, therefore, arranged for a WHEATSTONE'S bridge, two arms of which were S and R, the other two arms being 20 units—made up of the two 5 and 10-unit standards already described, while the fourth was the single coil G, The box was used as a shunt to the 20-unit arm, and adjusted as before till the galvanometer showed no deflection, the effective resistance of this arm was found to be 19·0444 units, while at the same temperature the value of G is ·99887 unit.

Substituting the value of R at the temperature of the observation and reducing our result to the temperature 12°, we find

$$S_0 = 3059{\cdot}37 \text{ units}$$

A second series of observations on a different occasion gave

$$S_0 = 3059{\cdot}86 \text{ B.A. units}$$

The mean is

$$S_0 = 3059{\cdot}62 \text{ B.A. units}$$

and this is the value we have used in our calculations.*

The coils were in all cases placed in water baths and allowed to stay for some hours in them. Before making the observations the temperature was read by thermometers graduated to fifths of a degree, which were compared with each other.

To determine then the value of the B.A. unit we must use the value of $R_0 = 160{\cdot}520$, found in this series of measurements. Now the values of S_0 and R_0 found from the boxes were respectively

$$3060{\cdot}80 \text{ and } 160{\cdot}586$$

These differ from the values we have obtained in our last observations by 1·18 and ·066 unit respectively, or rather more than 1 in 3000. This difference would correspond to an error of about 1° in the measurement of the temperature.

Considering then the uncertainty which must attach to the temperature of the coils inside the box, it seemed best to take our last values rather than the mean of the two —we would rather regard the first series as a check upon any large error. But though this difference in the value of S will affect to the amount of 1 in 3000 the value of R_0 in absolute units, it does not affect at all sensibly the value of the B.A. unit, for this latter depends on the ratio of S_0/R_0; taking the values of S_0 and R_0 from the boxes, we have

$$\frac{S_0}{R_0} = 19{\cdot}0602$$

while, if we use the last values obtained for S_0 and R_0, we get

$$\frac{S_0}{R_0} = 19{\cdot}0607$$

the difference is only about 1 in 40,000 and does not concern us. We shall therefore put $R_0 = 160{\cdot}520$ ohms, the temperature being 12°.

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* The value 3059·89 used on page 255 is obtained from this by applying a temperature correction to the 60 units, the resistance of the galvanometer, which was at 13°·5.

The contrivance designed by Lord RAYLEIGH by means of which our 24-unit standard was compared with the B.A. units needs a special reference: it has been described by Lord RAYLEIGH in his second paper on the value of the B.A. unit, an abstract of which was read before the Royal Society on March 9, 1882, while the paper is published in the Phil. Trans., Part I., 1882.

Five coils each of approximately 5 ohms resistance were wound and enclosed in a box, from which the two electrodes, copper rods with amalgamated ends, of each coil protrude. By means of two series of mercury cups this system could be put either in series or in multiple arc. Then, if each coil of the series is so nearly equal to 5 units that we may neglect the square of the difference, it is easy to show that the resistance of the system in series is exactly 25 times that which it has when in multiple arc. The coil of 24 units and a single unit were arranged in series so that they could readily be put into connexion with FLEMING's bridge. The set of five 5-unit coils in multiple arc was compared with a single unit. The connexions were rapidly altered, and the five 5 units in series were compared with the 24+1; then again adjusting the connexions, another comparison between the 25 units in multiple arc and the single was made.

In this manner a value of the 24 ohms was obtained in terms of the single ohm, the result of the comparison being as already stated.

Thus, whether we use as the values of R_0 and S_0 those found from comparison with the box or those determined by comparison with the coils, we have as values of the B.A. unit determined from this second part

Series A	·98673	mean of 4
„ B	·98598	„ 2
„ C	·98716	„ 3

While the mean result is

·986706 ohm.

The result obtained in June, 1881, Part I., as the mean of three complete sets, was

·986350 ohm.

Our discussion has shown us that the possible errors of this determination are considerably greater than that obtained in Part II. We will, therefore, give to each experiment in Part I. only half the weight of an experiment in Part II., and obtain thus as our final value for the ohm

·98665 ohm.

The value obtained by Lord RAYLEIGH in his second experiments with the rotating coil (Phil. Trans., Part I., 1882) is

·98651* ohm.

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* Since this paper was read Lord RAYLEIGH has obtained by two modifications of LORENZ' method the two values ·9867 and ·9868.

The difference, amounting as it does to about 14 parts in 100,000, is obviously less than the probable error of our result, and there can be little doubt but that the value ·9866 is only two or three parts in 10,000 from the truth. This value, as Lord RAYLEIGH has pointed out, is strikingly confirmed by JOULE's latest determinations of the mechanical equivalent of heat.

In conclusion, we would discuss further one or two possible sources of error. We have assumed that the ratio of V/S is independent of the temperature. This is not true, for S consists of two parts; one part of about 3000 units has the same temperature coefficient and is always at the same temperature as V, so that for it our assumption is justified; the other part of the galvanometer, of about 60 units, is of copper, of which the temperature coefficient is ·003, and it is not at the same temperature as V.

Now the whole range in the temperature of the galvanometer is about 3°, the range on either side of the mean 1°·5, and an error of 1°·5 in the temperature of the galvanometer we can show would affect R to the amount of about one part in 8000. Thus those values of R for which the temperature of the galvanometer differs most from the mean may be affected to the amount of 1 in 8000 by this source of error. Taking the mean of all our observations, however, the error vanishes.

Professor CHRYSTAL had warned us that he had had some difficulty in securing sufficiently good insulation between the wire of the coils A and B and the brass rings in which they were wound. We therefore tested both coils to see that this was maintained. One pole of a battery of 25 LECLANCHÉ cells was connected with the brass ring, while the other pole was put in contact with one electrode of a galvanometer of 2000 ohms resistance, the second electrode of the galvanometer being connected with the wire of the coil.

Deflections of 60 and 80 divisions of the scale were observed for the two coils A and B respectively. The same battery through 100,000 ohms when the galvanometer is shunted with 8 ohms gave a deflection of over 100 divisions, the insulation resistance therefore is considerably over

$$\frac{100,000 \times 2000}{8}$$

or

$$25,000,000 \text{ ohms.}$$

The error that might arise from the use of a paper scale has been discussed. In the calculations corrections have been applied to the scale readings to reduce them to the standard metre. The corrections taken were the mean of four series of observations—two by R. T. G., two by E. B. S.; and these, though made at very different times, varied only by quantities comparable with the error of an observation.

The most serious objection, however, that can be raised applies to all observations in which a ballistic galvanometer is used, and there were two points here which seemed to require special notice. Is it right to assume that on reversing the primary current the change that takes place is over in a time which is small compared with the time of swing?

Now our reversals were made with an ordinary rocking commutator dipping into mercury cups, and only occupied a small fraction of a second. We therefore made a series of experiments to see if we could find the effect produced by holding the primary circuit open for definite small periods.

On reversing in the usual rapid manner we obtained a throw which varied between 119·5 and 119·3 millims.

When our primary circuit was held open for one second the throws were 118·9, 119·3, 119·3, and 119·1, while if the contact was broken for two seconds the effect on the throw was marked. Thus if by chance in any experiment the primary circuit was open for as much as a second, so that the battery current took something over a second to get steady, it would only produce an effect of about 1 in 1000 in the result; we feel quite certain that in no case the period of break occupied more than a small fraction, from a quarter to one-sixth, of a second.

Again, it might happen that the somewhat powerful induced current passing through the galvanometer coils might alter—temporarily or permanently—the magnetic moment of our needle.

A permanent alteration would of course be indicated by variations both in the throw produced by the induction current and in the time of swing. In the actual experiments after the induction current had been passed in one direction through the coils of the galvanometer, and the throw observed, it was passed in the other direction to stop the vibrations of the needle, and it was possible that each current might have produced real permanent changes in the magnetic moment, but of exactly equal amount, so that we had observed no appreciable alterations in the time of swing which we could assign to this cause.

We therefore took a series of measurements of throws in which the current was only allowed to pass in one direction through the coils, the vibrations of the needle being stopped by means of an external damper; the times of swing also were observed at intervals.

The table below gives the result.

Time of swing.	Throw.
	115·8
23·383	
	116·0
	116·0
	115·7
	116·0
	115·9
23·389	

SECOND series.—Throws in opposite direction.

Time of swing.	Throw
	115·5
23·389	
	115·6
	116·2
	115·8
23·392	

The throws throughout are very nearly equal; the time of swing, it is true, has increased by ·009″, but this change cannot be due to an alteration in the magnetic moment of the needle, for if the current passed in the one direction had decreased the magnetic moment and so increased the time of swing, that passed in the other direction in the second series would have increased the magnetic moment and so decreased the time of swing. The alteration in T then we must rather ascribe to a variation in the value of the horizontal intensity. This has been allowed for in the experiments by the repeated observations which were made of T. As regards the temporary effect of the induced magnetism produced by the current while it lasts, we notice that its direction would at each instant be perpendicular to the plane of the coils, so that the force between the magnet thus formed by the current and the current forming it would be along the axis of the temporary magnet and have no tendency to cause it to move. In fact, the field of force produced by the coils being uniform, the actual force on the induced magnetism will be zero.

A number of observations were also made to see if the time taken by the battery current to become steady after a reversal was appreciable. The galvanometer circuit being open the battery circuit was reversed, and then the galvanometer circuit closed at an interval of from ·25″ to ·3″ after.

If after this interval the battery current had still been varying an induction current would have shown itself in the galvanometer, but though the reversals were made in three separate experiments, 13, 7, and 10 times respectively, in time with the period of the galvanometer, no effect was produced.

We are sure, therefore, that the battery current has attained its steady value in less than ·3″ after reversal, while our former experiments have shown us that if the period of change were as great as 1″ the error produced in the throw would be less than 1 in 1000.

The direct effects of the connecting wires and commutators on the galvanometer during the experiments were carefully tested for but no result could be found. Copper wire insulated with gutta-percha was used for the connexions, and the wires in each circuit were carefully twisted up together.

Most of the apparatus employed belongs to the Cavendish Laboratory, where the experiments were conducted. Our best thanks are due to Lord RAYLEIGH for his unceasing kindness and his many valuable suggestions which have helped us satisfactorily to surmount several serious difficulties, and have added greatly to the accuracy of the results.

NOTE.

(Added December 2, 1882.)

Another source of error should have been mentioned in the previous discussion.

The external diameter of each layer was measured necessarily before the layers external to it were wound on. There will probably be a tendency in each layer to compress those below it, and thus to make the mean radius of the coil somewhat less than we have assumed.

In fact, Lord RAYLEIGH found in his experiments (Phil. Trans., Part I., 1882) that the value of the mean radius obtained from measurements made as the coil was unwound was less by 1 in 2000 than that obtained from the measurements made while winding.

Several reasons lead us to believe that any effect of the kind would be much less with our coils.

The error is produced probably by the silk being pressed over the top and bottom of the layer into the intervals between the wires—the wires being circular in section there is necessarily a space between them.

Now the amount of this free space would be proportional to the area of the cross section of the wire, while the amount of silk would depend on the circumference of this cross section.

The diameter of our wire was only about two-thirds of that used by Lord RAYLEIGH, there would, therefore, be less space in proportion to the amount of silk into which the silk could be squeezed, and the error produced would be less.

Again, if this yielding is going on, it is clear that the average thickness of a layer should appear to decrease as we get near the outside surface. A reference to the measurements shows that this is not the case. On dividing the whole series of layers into sets of five, and taking the average for each set, omitting a layer in which paraffin paper had been used, we find the values for the average thickness ·128, ·123, ·127, ·128, and ·127.

No gradual decrease is here observable. We therefore feel confident that the error was much less than in the coils wound by Lord RAYLEIGH.

A strong indirect confirmation of the accuracy of the value of the mean radius used is afforded by Lord RAYLEIGH's recent measurements by LORENZ' method.

The same coils were used as in our observations, and Lord RAYLEIGH has shown that an error in the mean radius would affect the two methods by about the same amount but in opposite directions. If δM be the error in M produced by an error δa in a the mean radius, it follows that in our case $\frac{\delta M}{M}$ is about $2\frac{\delta a}{a}$, while in LORENZ' method, M being the induction between the disc and the coils, for the arrangement adopted $\frac{\delta M}{M} = -2\frac{\delta a}{a}$ approximately—in the one case δM is of the same sign as δa, in the other the signs are opposite. The mean of the results then obtained from the two methods will be free from an error in a. But the two results are ·98665 and ·9867. They are identical to the fourth figure. Thus we infer that the error in a is probably very small.

V. *On the Normal Paraffins.*—Part IV.

By C. Schorlemmer, *F.R.S.*, *and* T. E. Thorpe, *F.R.S.*

Received December 5,—Read December 21, 1882.

In the last communication made by one of us on this subject (Phil. Trans., 1880, p. 451), it was stated that we contemplated making a joint investigation on the heptane from *Pinus Sabiniana*. A large quantity of the pure heptane was treated with chlorine in the manner described in this series of communications (Phil. Trans., 1872, p. 111), and the chlorides boiling between 143° and 157°·5 were converted into the alcohols in the ordinary way. The primary alcohol boiled between 165°—170°, whilst the secondary alcohol distilled over, for the most part, between 156°--158°. The alcohols were then oxidised in the manner described in Phil. Trans., 1872, p. 121. The ketone obtained from the secondary alcohol was further oxidised by being heated with the chromic acid solution in sealed tubes at 100°. The liquid was then neutralised and the unattacked ketone was separated by distillation. In order to isolate the acids contained in the residue small quantities of sulphuric acid were added, and the acid which separated out (pentoic acid) was distilled off in a current of steam—the operation being repeated until the pentoic acid was no longer recognisable by its smell. The acetic acid contained in the residue was then separated by a further addition of sulphuric acid and distilled off by direct heating. The respective silver salts of the two acids were then prepared from the several fractions in the ordinary way. Analysis showed that the separation of the acids was complete.

I. Silver salt from the normal alcohol.

(*a*) 0·4035 grm. salt gave 0·1842 grm. Ag=45·6 per cent.
(*b*) 0·210 „ „ 0·0953 „ =45·4 „

Calculated for silver heptoate 45·5 per cent.

II. Silver salts from ketone from secondary alcohol.

0·2022 grm. salt A gave 0·1050 grm. Ag=51·9 per cent.
0·1491 „ „ 0·0771 „ =51·7 „

Silver pentoate =51·6 Ag.

0·235 grm. salt B. gave 0·1518 grm. Ag=64·59 per cent.

Silver acetate =64·67 per cent.

The results of the oxidation show therefore that primary heptyl alcohol and methyl-pentyl-carbinol had been formed, as in the case of heptane from petroleum.

Another portion of the mixture of the monochlorides was then heated to 100° with alcoholic solution of potash, whereby in addition to heptylene a mixture of the ethyl heptyl ethers was obtained. The heptylene, purified by repeated rectification over sodium, boiled constantly at 98°·5. It was placed in contact with an excess of fuming hydrochloric acid in the dark and in a well-closed bottle for six weeks. Heptylene from petroleum heptane was similarly treated.

Now it is remarkable that whilst hexylene from mannite combines completely with hydrochloric acid under these circumstances forming secondary hexyl chloride (Phil. Trans., 1880, p. 457), the greater part of the heptylene from *Pinus* heptane was found to be unattacked, not more than 10 per cent. of heptyl chloride having been formed. On the other hand, about one-half of the heptylene from petroleum had been converted into the chloride in accordance with the former observations made by one of us on this point.

The uncombined portions of both specimens of heptylene were again placed in contact with the fuming acid, and (the research being interrupted by other work) they were allowed so to remain for many months. At the expiration of this time it was found that the *Pinus* heptylene had united almost completely with the acid, whilst an additional quantity of the heptylene from petroleum had likewise entered into combination. It follows from this that, contrary to expectation, hydrochloric acid acting in the cold is not capable of effecting the separation of isomeric paraffins (Phil. Trans., 1880, p. 451).

It is remarkable that the *Pinus* heptylene should require so long a time to bring about its union with hydrochloric acid, since, as will be shown immediately, its constitution is exactly analogous to that of the hexylene from mannite or propyl-methyl ethylene, $C_3H_7.CH{=}CH.CH_3$, which so easily goes into combination.

In order to establish the constitution of the heptylene from *Pinus* it was oxidised by means of a solution of potassium dichromate in dilute sulphuric acid in the manner adopted by HECHT in the oxidation of hexylene from mannite (Ber. Deutsch. Chem. Ges., Bd. xi., S. 1152). The acids so formed were converted into the silver salts and analysed with the following results:—

Fraction.	Salt taken	Silver found.	Silver pentoate 51·67.
1	0·0855	0·0443	51·81
2	0·0560	0·02875	51·34
3	0·1140	0·0590	51·75
4	0·1350	0·0700	51·84
5	0·0685	0·0350	51·09
6	0·1000	0·0540	54·00
7	0·2200	0·1370	62·27
8	0·1370	0·0855	64·60
9	0·1400	0·0910	65·00
			Silver acetate 64 67,

The first five fractions consisted of a fine crystalline powder, the last three were glistening needles.

These observations clearly indicate that only pentoic and acetic acids had been formed by the oxidation, and therefore that the *Pinus* heptylene is pure butyl-methyl-ethylene, $C_4H_9.CH{=}CH.CH_3$, which had been formed from the secondary heptyl chloride, $C_4H_9{-}CH_2{-}CHCl.CH_3$.

These results, taken in conjunction with those of former investigations, leave no doubt, therefore, that by the action of chlorine upon a normal paraffin not all the chlorides indicated by theory are formed, but only the primary and a secondary chloride which contains the group $-CHCl.CH_3$. One of us has formerly shown that by the action of bromine upon normal paraffins from petroleum only secondary bromides corresponding to chlorides are formed (Phil. Trans., 1878, Part I., p. 49). VENABLE has since shown that *Pinus* heptane is acted upon in the same way (Ber. Deutsch. Chem. Ges., Bd. xiii., S. 1649).

VI. *On a Collection of Rock Specimens from the Island of Socotra.*

By T. G. BONNEY, *M.A., F.R.S., F.G.S., Professor of Geology in University College, London, and Fellow of St. John's College, Cambridge.*

Received June 12,—Read June 15, 1882.

[PLATES 6–7.]

THE specimens of rocks—about 500 in number—collected by Professor BAYLEY BALFOUR during his late exploration of the Island of Socotra, were forwarded to me for examination. Several of these specimens, as was to be expected under the circumstances, were in a condition unfavourable for precise determination, being often fragments from weathered surfaces and sometimes much decomposed. Each, however, has been described as far as the circumstances would admit, and microscopic slides have been prepared for me by Mr. CUTTELL from about 80 of the more interesting.

As it happens, certain of these offer difficulties which in the present state of our knowledge are almost insuperable. While the use of the microscope has dispelled much confusion in our petrological ideas and supplied us in many respects with a firm basis of knowledge, it has not in every case—owing to the novelty of this mode of research and the inherent difficulties—enabled the student to feel perfect confidence in some of his conclusions, especially when he is restricted to this method of examination. Perhaps the greatest of our petrological difficulties is the distinguishing in every case between certain highly metamorphosed rocks and those of similar chemical composition which are truly igneous. It has, indeed, been maintained by some eminent geologists that certain sedimentary materials may be so altered by the combined action of water, heat, and pressure as to be converted *in situ* into a rock indistinguishable from one of those commonly held to be of igneous origin. Accordingly we read not seldom of "metamorphic granite" and of "gneiss passing into granite," for it is of these that the above opinion is commonly held. Other geologists, indeed, go yet further and make a similar assertion, not only of the more coarsely crystalline rocks such as syenite, diorite, and gabbro, but even of the more compact varieties of felstone and greenstone, which in like manner are said to afford indubitable examples of transition into beds of true sedimentary origin. As regards this view, we may admit that if a mass of clastic materials be once reduced to a molten magma its past history is obliterated; and, further, that there is no reason, so far as we know, why this melting down should not occur. In this sense, any igneous rock whatever may possibly deserve the name of metamorphic. But, while admitting the *à priori* possibility of such a change, I must confess to being sceptical as to whether any case of it has yet been fully proved to exist. I have studied not a few of the alleged instances patiently and, as I trust, without prejudice, always with the following results—either that there was no evidence which was conclusive on either side, or that to a practised eye there was very clear evidence against the asserted transition—*i.e.*, that (to refer to the instance named above) either the so-called granite was clearly only a granitoid gneiss or else that it was distinctly intrusive in the schistose rock. In these investigations the microscope is a very great help, but I freely admit that there are many cases where we cannot rely upon it alone, and must also study the rock in the field. Our knowledge at present does not enable us to pronounce upon the classificatory value of certain structures which we observe in the microscopic study of some specimens. This difficulty, however, is one which time and experience will probably remove. To admit the existence of cases where it is at present safer to suspend the judgment in no way concedes that it is impossible ultimately to arrive at a conclusion. So then, while not professing in every case from examination of hand specimens alone to decide whether a rock is a granitoid gneiss or a true granite, I believe in the distinctness of the two rocks. As it happens, some of the specimens from Socotra belong to this dubious class; and I cannot say positively, even after microscopic examination, whether in certain cases we have a granitoid gneiss or a true granite, and in others a hornblendic rock of sedimentary origin or a true diorite.

In drawing up my report on Professor BALFOUR's collection I have thought it best to give, first, a general description of the petrology of those parts of the island over which he travelled, and then to describe the more important varieties of each group of rocks with which his journey has furnished us.[*]

The island of Socotra is about 72 miles long from east to west and about 22 miles in breadth. The general physical features are thus described by Professor BALFOUR in his account of the island printed in the volume of reports of the British Association for the year 1881 :—

"The surface features of Socotra at the present time are those of an island mountainous in the extreme. The shore line on its southern aspect is, as the map shows, a tolerably continuous one, unbroken by deep inlets or bays. On the northern side occur a few shallow bays at the mouths of the streams, which afford the only anchorage to be obtained around the island, but no one of them is safe at all seasons of the year. On all sides the hills rise with considerable abruptness over a wide area, forming bold perpendicular cliffs of several hundred feet in height, whose base is washed by the waters of the Indian Ocean, but at other places leaving plains varying in breadth up to as much as five miles between their base and the shore. On the south side of the island is the largest of these shore plains (Nogad), which, extending nearly the whole length of the island, is for miles covered with dunes of blown sand. On the north these plains occur chiefly at the mouths of the streams, and are the sites of the only places which may be called towns.

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[*] In writing the first draft of this paper I followed the other plan, and described the specimens as they were collected by him during each section (generally representing a day) of his journey; but as I found this involve prolixity and needless repetition, I have recast the paper into its present form.

"The internal hilly part of the island may be roughly and shortly described as a wide undulating and intersected limestone plateau of an altitude averaging 1,000 feet, which flanks on the west, south, and east a nucleus of granitic peaks approaching 4,000 feet high. The whole of this hilly region is deeply cut into by ravines and valleys. These in the rainy season are occupied by roaring torrents, but the majority of them remain empty during the dry season. There are, however, many perennial streams on the island, especially in the central granitic region, where amongst the hills the most charming bubbling streams, dashing over boulders in a series of cascades or purling gently over a pebbly shingle, make it hard to believe that one is in such proximity to the desert region of Arabia. Few of the perennial streams reach the shore in the dry season—most of them are fiumaras.

"The eastern end of the island is most destitute of water. Here in the dry season are no rivers, and, springs being rare, it is the most arid region.

"Igneous, metamorphic, and sedimentary rocks compose the island. The fundamental rock is granitic. This crops out, as I have mentioned, towards the middle of the island, forming a series of bare pinnacles and crags, projecting, with singularly fantastic look, from the plateau below. This rock also shows on the slopes of the valleys and ravines below the compact limestone which caps it and forms the surface rock of the hill plateaux. This limestone attains in places, as seen on the cliff faces, a thickness of two or three hundred feet. Superficially, over wide areas it is rotted and broken into a jagged surface, over which progression is by no means easy, while at other spots it forms broad, smooth slabs. A shaly rock and coarse-textured purple sandstone, in beds dipping at all angles, crop out in the valleys and on the shore, whilst forming the shore-plains and the bases of the valleys is a recent breccia and conglomerate. Cutting through all these rocks, and altering them to a considerable extent, occur dykes and extensive masses of doleritic rocks and felstones, which vary much in texture."

Professor BALFOUR landed on the shore of Gubbet Gollonsir at the north-western extremity of the island and explored the district fringing this bay for some distance inland, so as to examine the high ground on either side of the level valley which opens out to the sea. The uplands here consist of a thick deposit of limestone, probably of middle tertiary age, but in many cases remarkably compact and hard, so that it has at times considerable resemblance to the well-known dolomites of the Italian Tyrol. This mass of limestone rests unconformably upon a group of highly crystalline gneisses, associated with diorites and perhaps with hornblende schists, which in general character correspond with the Hebridean series in the north-west of Scotland. These older rocks are frequently exposed in the beds of valleys and in the lower part of the walls of the plateau. The same description applies to the elevated districts traversed by Professor

BALFOUR on his way to the shore in Kurmeh Bay; but judging from the specimens which he has brought, I think it probable that some true granite is also associated with the gneissic series. The limestone, supported as usual by the latter, extends along the shore of Kurmeh Bay, and towards the eastern part it is cut (together, of course, with the older rocks) by basalt dykes. Felsite also occurs somewhere in this district, probably inland towards the south-east.

Further east along this plain, in the vicinity of Kadhab village, basalt dykes are seen cutting the limestone rock, and there are scattered boulders of granite or granitoid rock, of red felsites and rhyolites, and of a hornblendic diabase. The cliffs by the sea between Kadhab village and Hadibu Plain are composed chiefly of the usual massive limestone, which rests on a green flinty shale or argillite. The lower part of the limestone is a breccia, in which are fragments of the subjacent rock, and bands in it even resemble the latter (possibly being composed of the same material redeposited). In this neighbourhood occurs a mass of a kind of kersantite, the presence of which may account for the peculiar "baked" aspect of the shale.

Similar argillites occur along the line of strike in the valley to the south-east of Tamarida village, overlain as before by the limestone. They have a rather sharp dip to the south-south-west, and "an imperfect vertical cleavage, or possibly, jointing." Apparently beneath, but possibly intrusive in the argillites, are some compact pale grey felsites.

Inland rises the fine Haggier range whose outlines somewhat remind us of the peaks of Sinai. The bulk of the specimens from this place are reddish or pinkish felspathic granites, not generally very rich in quartz and always poor in the third constituent, generally rather coarsely crystalline, to which we may give the name of pegmatite.* This is cut by veins of a more compact granite and by at least one dyke of mica-trap. It is possible that gneiss also is present in this region. About the margin of the range the limestone rests upon the granite, and contains detritus from the latter in its lower part. The approximately highest point of the range is a pegmatite. On the south side about Adona we have the same pegmatites with a more hornblendic granite, and intrusive dykes or veins of compact purplish quartz-felsite or rhyolite.

Eastwards from Hadibu, on the way to Jebel Omhari, blocks of reddish felsite or rhyolite are shown on the plain, and the limestone is presently reached. The extreme north-west of the Girgha range is formed of pegmatites similar to those of the Haggier mountain, varying from coarse to fine in texture, and of a rather compact quartz-felsite; the latter constitutes a considerable hill mass, and is probably intrusive in the former. There is also some intrusive diabase.

Between Kami and the hamlet Ma-aber, on the Motaha, the rocks vary considerably. Granites of the usual type, compact reddish or greyish quartz-felsite or rhyolite alternate one with another. A mica-trap is also found. Evidently this is an extension of the last group of rocks. A specimen of an epidotic quartzite proves that there are, at any rate, some indications of the metamorphic series.

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* There is some authority for applying this term (often used vaguely) to granites which consist almost wholly of quartz and felspar. Of these, "graphic granite" is a variety.

Following the Motaha river in a south-south-west direction we find pegmatites, coarse and fine-grained felsites, red, buff, and grey in colour, together with a greenstone so much decomposed that it is difficult to decide whether it be augitic or hornblendic. The specimens prove that the typical rocks of the Haggier range extend at least thus far to the east.

Returning to Ma-aber and continuing the section eastwards for three or four miles to Gedidery, Professor BALFOUR found first more red felsite, and then, near the latter place, much variation in the character and the colour of the rocks—"grey, white, yellowish, and red beds alternating." The specimens from these show that the following rocks are represented · fine-grained granite, compact felsites, diorite or, possibly, hornblendic schist and decomposed greenstone. It is also seen that the felsite, like the granite, is overlain by the usual limestone.

Between Gedidery and Gharrieh fine-grained granites and compact quartz felsites were collected, indicating that the Haggier group is prolonged as far as Khor Gharrieh, where these rocks come down to the sea. The granitic series was traced inland up the course of a stream, flowing generally north-north-west as far as the hamlet Vishas, and is cut by diorites. From this place up the Goahal Valley, in an easterly direction, crystalline rocks may be traced beneath the limestone, but some of the specimens must, I think, be gneisses, so that the ancient metamorphic series probably occurs in the eastern as well as in the western part of Socotra. Probably it could be traced still further east, but Professor BALFOUR, owing to indisposition, was unable to collect any specimens for the next seven miles. The limestone plateau stretches away to the eastern extremity of the island, specimens of this rock being brought from Ras Mom, a hill at the neck of the long terminal peninsula, which was the turning point of the expedition.

After retracing his steps for a few miles to the west, Professor BALFOUR travelled for some distance over the limestone plateau as far as Kittim, shortly after leaving which place he turned to the north-north-west, passing first over considerable masses of rhyolitic breccias, conglomerates, and grits. A specimen was also brought from a vein of calcite, containing fragments of red rhyolite and pinkish felspathic granite. Some of these fragments appear to be of volcanic origin, and there are masses of compact quartz-felsite, and of rhyolite with distinct fluidal structure, which, as Professor BALFOUR remarked, appeared very like lava flows. All these rocks are overlain by the plateau limestones.

From Azorah Professor BALFOUR turned in a west-south-west direction, following the Mitgahon gorge down to Baha, a village lying at the base of a peak called Töf, the easternmost summit of the Haggier range. Very compact quartz-felsites, and rhyolites abound. They form a conspicuous hill called Afero, which appeared to him to resemble the neck or "core" of a volcano. Near the opening of the gorge are doleritic rocks, which extend for some distance, and are, at any rate in some cases, intrusive in the granitic series which is sometimes fine-grained and is more hornblendic than usual. The red felsite also occurs.

After returning to Hadibu Professor BALFOUR crossed the Haggier range, and travelling in a general southerly direction reached the Nogad plain about the middle of the south coast of Socotra. But few specimens were collected during this part of the expedition, and the only two of interest were a red rhyolite, near Feraighey, and a rather peculiar dolerite, which will be more minutely described under their proper heads. On the plain were loose blocks of a hard conglomerate containing well-rounded pebbles (up to about 1 inch or longer in diameter) of compact red quartz-felsite and purplish rhyolite with well marked fluidal structure embedded in a silicious paste. Besides these, blocks were found of white cherty fragments, also in a silicious matrix, which contains grains of the above igneous rocks. After leaving the Nogad plain Professor BALFOUR again crossed the limestone plateau to the north coast near Gubbet Kadhab, whence he returned to Hadibu.

I now proceed to describe the more conspicuous and interesting varieties of rocks in Professor BALFOUR's collection, taking the more crystalline and most ancient series first in order.

Gneissic rocks.

The existence of a series of highly crystalline metamorphic rocks in the Island of Socotra is indubitable, notwithstanding the difficulty already mentioned, of deciding without further examination whether certain of the specimens are to be referred to the granitoid gneisses or the granites. This series forms the floor on which rest the great limestone plateau and some other sedimentary deposits, and is well exposed in the western and to some extent in the eastern part of the island, being replaced in the central and most elevated region by true granites. The following are the principal varieties: (*a*) gneisses, not conspicuously foliated, consisting of quartz, felspar (generally of pinkish colour) and mica, black and white, and possibly hornblende. As a type of these a specimen from a hill near the opening of Gollonsir Valley was examined microscopically. The slide exhibits quartz, felspar, little elongated clusters of green hornblende with some epidote, opacite, ferrite, and a few scales of white mica. The felspar is chiefly microdine, a variety which is remarkably abundant in the Hebridean gneisses and in others of great antiquity. There is also a plagioclase, probably albite,* and some of the felspar crystals contain groups of small colourless microliths with a parallel arrangement. This structure I have also observed in some of the old gneisses from Greenland. The quartz contains small cavities (rarely with bubbles), prismatic microliths of very pale green colour, and numerous colourless hair-like microliths, scattered irregularly throughout the grain. The resemblance of the specimens to examples from the Hebridean series of Scotland, the Laurentian series of America, and other Archæan rocks is very striking. [4115], one of a large series from the right bank of the upper part of the Gollonsir Valley, contains a fair proportion of black mica instead of hornblende. Apatite can be recognised, and the small, almost colourless, prismatic microliths already mentioned, some of which may be referred to this mineral, though I am of opinion that the majority are different. There is no microcline, but a good deal of plagioclase, probably both albite and oligoclase. [4124] contains hornblende with a black mica and sphene. (*b*) granitoid gneisses, consisting mainly of quartz and felspar, belonging to the group for which I have proposed the name granitoidite.* In these the quartz and felspar correspond with those described above, and the principal difference is that the rock is mainly composed of these two minerals, with an occasional grain of iron peroxide and a flake or two of iron glance or a ferruginous mica, hornblende or epidote. (*c*) hornblendic rocks, which we shall notice in the next paragraph, and (*d*) an impure quartzite from near Kami, north-east of the Haggier range. This rock consists of quartz, decomposed felspar, epidote, and perhaps a little hornblende, with, in parts, a good deal of magnetite.

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* In attempting to determine the species of the felspar I have made use of the optical tests given by Professors FOUQUÉ and LÉVY (among others) in their magnificent work "Roches Eruptives Françaises." I cannot, however, say that I feel great confidence in the results of the method.

Diorite and other hornblendic rocks.

I have made this division somewhat vague for two reasons: one that in rocks of igneous origin the hornblende is not unfrequently of secondary origin, having replaced augite or diallage, so that the rock is more properly a uralitic or hornblendic diabase than a true diorite: the other, that the means of making a thorough study of some difficult examples of these rocks—corresponding with those already mentioned in the gneisses and granites—are only now being obtained by me.† From the hill near the opening of Gollonsir Valley is a series of specimens, which, according to Professor Balfour, come from what appears to be a dyke in the Archæan series. Some are coarser in texture than others. Two of these [4009] and [4005] have been examined microscopically, of which the former was *in situ*, the latter from a loose block. They consist of a decomposed felspar, in which, however, the remains of the twinning characteristic of plagioclase can occasionally be discerned, of hornblende, black mica, and opacite, with some apatite, and a few grains of epidote. The hornblende is green in colour and exhibits very characteristic cleavage. The mica is sometimes altered to a greenish mineral, and often contains needles and grains of opacite, generally arranged parallel with the principal cleavage planes. In [4053] "from the slope of the hill above Hanigon, to the west of the former locality," the felspar is less altered, and some of it, if we may trust the optical tests, is albite, and more of the green mineral appears to be an altered mica. Among the Archæan rocks on the right bank of the upper part of the Gollonsir Valley are two dykes. One of these [4145], a dark speckled rock with the felspathic constituent weathering yellow, consists of long narrow crystals of rather decomposed plagioclastic felspar and a hornblendic mineral; some of this shows very characteristic cleavage, other crystals when cut parallel with the edges of the prism exhibit a rather fibrous structure and look as if they might be of secondary origin. One or two possibly replace magnesia-mica. There is a little quartz, epidote, and some black iron peroxide, with, perhaps, a little apatite. The other dyke has much of its hornblende in long narrow crystals with the same secondary aspect, and the felspar crystals also, which to a great extent are replaced by secondary microliths, are long and narrow like those of a dolerite. A third specimen [4197] from a rock on the bank of a stream is of the same character, but rather more porphyritic. Many of the felspar crystals show a zonal structure, and the outer part remains clear, while the inner is replaced by secondary microliths. The small extinction angle, where observable, leads me to conclude that oligoclase is the dominant felspar. In this slide magnesia-mica is recognisable.

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* Quart. Jour. Geol. Soc., vol. xxxv., p. 322.

† It must be remembered, in excuse for this ignorance, that it is of no use to purchase specimens for study of these difficult cases, or, as a rule, to trust the statements which one finds in print. The student must collect his own specimens, and to do this it is necessary to visit distant localities and expend much time and money. Hence difficulties are but slowly removed.

From the tracts traversed on the east of Ghubbet Kurmeh a remarkable rock has been brought, which, with some hesitation, I class among the diorites. The compact matrix is a dark blue, almost black colour, in which are thin crystals of felspar often nearly 1″ long. With the microscope the ground mass is seen to consist of small elongated felspar crystals having a rather fibrous or "teazed" aspect, associated with much opacite and some green microliths of hornblende or chlorite, rather irregularly disseminated, and some flakes of brown mica. The larger crystals of plagioclase felspar show the greatest absorption at very small angles with the vibration planes of the crossed nicols, and parallel lamellæ extinguish almost simultaneously, so that they are probably oligoclase. The hornblende, however, has very much the aspect of a secondary product, and I am disposed to regard the rock rather as a hornblendic diabase than a true diorite.

From an isolated hill in the middle of the Gharrieh Plain (mainly limestone), which is covered with a peculiar vegetation, come some dioritic rocks (associated apparently with granites or granitoid gneisses). One of these consists of a plagioclastic felspar, which (though in part replaced by microliths) from its large angle, where unaltered, between successive extinctions (often quite 30°) is probably, in part at least, anorthite, and of hornblende, also with a secondary aspect, some chlorite, with epidote, apatite, and a dichroic fibrous lamellar green mineral, which extinguishes parallel with the lamellæ, and is more probably an altered mica than hornblende. A few grains of quartz are also present.

Other hornblendic rocks were noted during my examination of the specimens. Those described above certainly appear to be in all cases igneous rock, but it is quite possible that among them may have been some of metamorphic origin referable to the Archæan series, for, as my experience in Cornwall and Scotland has shown me, it is sometimes by no means easy to distinguish the one from the other in hand specimens.

Dolerites, basalts, &c.

The information furnished to me does not allow me to attempt any classification of this group. Some of the basalts cut the limestones and so are among the most modern known igneous rocks of the island, but whether all these augitic rocks are of the same date it is impossible to say: probably they are not.

A dolerite from the plain on the south-west of the stream in Gollonsir Valley [4051], is a rather coarsely crystalline rock, consisting mainly of whitish felspar and black augite. The former, under the microscope, proves to be well preserved labradorite, the crystals of which have fairly regular linear boundaries, and in form are rather broad oblongs. The augite, which has solidified after the felspar, is full, in one or two cases, of irregular grains of opacite, which sometimes make up almost the whole of the crystal. Incipient conversion into uralite is exhibited, crystals of augite having sometimes an irregular border of the latter mineral; while sometimes minute scales of it are disseminated in the nearly colourless augite crystal. In one or two instances almost the whole crystal is replaced by uralite.

One of the dykes near Kadhab village [4198], I leave, after microscopic examination, with some hesitation among the basalts. It retains some traces of a clear glassy base crowded with opacite and other microliths, some acicular and colourless, which with some larger crystals are rather like anorthite, others are of a greenish-yellow colour, fibrous or filmy; these being generally associated in irregular patches, with interspersed specks of opacite. They are most like replacements of an augitic constituent. A little hornblende can also be recognised. There are some indications of fluidal structure, and the rock evidently approaches the augite-andesite group.

[4454], one of a group of rocks of doleritic aspect, from the base of Azalin on the bank of the Hasainho, speckled whitish and blackish, consists of (*a*) labradorite in well defined crystals, evidently the first mineral to consolidate; (*b*) a slightly brown, rather dichroic augite, which in one case approaches diallage in its close cleavage; (*c*) a dark brown hornblende, sometimes inclining at the edges to sap-green (Plate 7, fig. 1). This paragenesis—for I think the whole aspect of the hornblende forbids the idea that it is of secondary formation (the uralites being generally green, and often quite pale)—though not very common is far from being unprecedented. It seems, for example, rather frequently in the old diabases of North Wales, and in the gabbro of Mont Colon (Pennine Alps). [4449], one of a very similar group of specimens, from the "rocks of which Azalin is composed," consists of well crystallized labradorite in good preservation, olivine, sometimes partially replaced by serpentinous microliths, opacite, augite, with a little diallage and hornblende. Some of the last mineral is certainly of secondary origin, bordering augite crystals, but one or two grains of darker colour may possibly be original constituents. The latter specimen then comes near to the true dolerites, the former to the Labrador-diorites.

From boulders in the bed of a stream-course near Fernaighey (no rock occurring *in situ*), come some specimens, of which [4499] may be taken as a type. The ground mass consists of a mass of small crystals of a white felspar and a black mineral, in which are scattered larger felspar crystals, sometimes 2 inches long. Under the microscope the rock is seen to consist of a rather decomposed labradorite, of a dull green, dirty-looking mineral associated with and occasionally replaced by opacite, probably a decomposed augite or diallage, and of a fibrous or scaly green mineral, giving bright tints with crossed nicols, also associated with grains of opacite. These aggregates have probably replaced olivine. There are also some fair sized crystals of apatite. The rock then has been an olivine-dolerite, but would now be more properly classed with the diabases.

Granites.

For the reasons given above I am unable to decide whether some of the specimens brought by Professor BALFOUR are true granites or only exceptional examples of the granitoid gneisses or granitoidites.* At the same time we need not hesitate to admit the existence of a considerable amount of this rock, especially in the more central part of the island, as for instance in the Haggier range. The granites of Socotra, as a rule, consist mainly of quartz and felspar, and so belong, as we have said, to the variety pegmatite. The latter mineral usually predominates and is commonly of a reddish colour. They are frequently much decomposed and not seldom show some indications of a graphic structure. It will suffice to select two or three from a large series as examples. [4249], "from the high cliff on the northern face of the Haggier range, one of the specimens typical of the rock forming the highest peaks," is a good example of the coarse red and white pegmatite. Microscopically it consists of a decomposed felspar, most, if not all, being orthoclase, quartz with numerous microlithic enclosures and minute fluid cavities, and a little of a dark green strongly dichroic mineral. Some of this is hornblende, but part of it much resembles tourmaline, to which mineral some acicular microliths, in one case in a tufted group, almost certainly belong. A second [4241] "from the slopes above Hadibu," has among it felspar, microcline, and a fair quantity of plagioclase (? oligoclase), grains of magnetite and a green chloritic mineral associated with clots of opacite, evidently replacing a magnesia-mica, associated with which is a little of a colourless mineral, possibly apatite. [4264], from the same region, is a "vein-granite, which cuts the coarser variety." It is a very finely crystalline quartz-felspar rock of a warm grey colour, containing some dark crystals about 0·25″ long. With the microscope it is seen to consist of intercrystallised quartz and felspar, the latter often exhibiting close twinning and sometimes being probably microcline.

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* Quart. Jour., Geo. Soc., vol. xxxv., p. 322.

The crystals occasionally exhibit rectilinear boundaries, but often interlace one with another and with the quartz in a very complicated way, as is not seldom the case in vein-granites. The slide contains some brown mica and tourmaline, the latter certainly a pseudomorph after the former. Two crystals retain a cleavage resembling hornblende, but behave optically as a uniaxial mineral, and when properly placed absorb the polarised beam far more completely than is usual with hornblende—a curious case of pseudomorphism, which, however, is not without parallels (Plate 7, fig. 2). [4205], an erratic near Kadhab village, doubtless from the western part of the Haggier range, is a red felspar-granite rather similar to the last, with a little iron-mica. [4207], a rather similar rock from the same region, exhibits in part a distinct micrographic structure; the decomposed felspar crystals are intimately interbanded with quartz, possibly of secondary origin; these two are most likely vein-granites. [4439], near Hesainho, in a region cut by felstone dykes, is a fine-grained pink and white granite with green specks, which under the microscope shows intercrystallised quartz and decomposed felspar (orthoclase and (?) oligoclase), with an approach to a graphic structure, besides some altered biotite—the rock is cut by a dyke of gabbro. Between these two rocks is a thin zone composed of the granite and the gabbro crushed, and to some extent mingled, though the materials of the former predominate. The constituents have been subsequently recemented, probably by deposition of quartz; this crushing was doubtless subsequent to the solidification of the newer rock. From the north-west of the Girgha range comes a series of specimens varying from coarse to moderately fine crystalline, and consisting of quartz and felspar with small quantities of hornblende and black mica, having a general resemblance to those already described from the Haggier range. Microscopic examination of one [4335] shows that the felspar is orthoclase, and perhaps microcline, with oligoclase. Small quantities of a chloritic mineral and a few microliths, possibly of tourmaline, are also present. The structure of the rock is rather abnormal for a granite, having some resemblance to the granitoid gneisses, but as there has been some local crushing and recementation, it may, notwithstanding, be a true granite.

Felstones and rhyolites.

Under this head I have retained a large number of rocks, which in some cases it would have been easy, but in others impossible, to subdivide. Petrologists are at present hardly in a position to agree upon precise definitions for the names of certain of the more acid igneous rocks, or upon the classificatory value of some of their minute structures. For example, a felstone must have a ground mass which is either microcrystalline or cryptocrystalline. But at present we cannot say whether or not these two structures correspond with different conditions in the past history of the rock, and so are of specific rather than of varietal value. Again, the latter structure, in some cases, appears to have been produced during the solidification of the mass, in others long afterwards. These, at present, it is often impossible to distinguish. Again, although we can often make a reasonable conjecture, we have no sure criteria for distinguishing between specimens taken from a dyke and from a flow, if they happen to have solidified under approximately similar circumstances. Hence, seeing that while in the present extensive series from Socotra there are some which are certainly from intrusive masses and others which are almost as certainly from lava flows, there are several which I cannot separate, either lithologically or petrologically, I have thought it better to group them under one general title, and indicate in the description what seems the most probable history of the rock from which the specimen has been taken. The majority of these Socotra felstones are of a warm red or purplish colour, though some of a grey tint also occur. They vary from minutely crystalline (approaching on this side vein-granites) to very compact subvitreous rocks which sometimes show a well marked fluidal structure. It will, perhaps, be more convenient to describe them as they occur geographically rather than to attempt to make a lithological arrangement.

The first specimen [4049] is from a dyke in the Archæan series, not far from the coast of Gubbet Gollonsir, about 1½ mile east of Ras Bedu, at the western extremity of the northern coast of Socotra. It is a compact dull-coloured felsite, weathering a pale yellowish-brown, containing many little black crystals. Under the microscope the rock appears to have a glassy base, stained with ferrite so as to present a rather muddy aspect, in which are scattered many small and well-formed crystals, both of felspar and green hornblende; among the former orthoclase and a plagioclase, possibly albite, can be recognised; the latter crystals are generally well formed but sometimes include portions of the ground mass. This, when examined with the two nicols, is to be seen full of minute microliths of felspar, but there appears to be some remains of a glassy base. There is no free quartz. The rock then appears to be intermediate between the hornblende andesites and the sanidine (or orthoclase) trachytes. From the same neighbourhood also comes a rather granular quartz-felsite containing some hornblende. Several specimens have been examined from an interesting series obtained on the plain near Kadhab village; unfortunately none of these occur *in situ*, but, as Professor Balfour informs me, they doubtless come from a part of the Haggier range lying to the south-east. [4206], a compact dark felsite with paler spots and wavy bands of a more crystalline material, containing small scattered crystals of pink felspar, exhibits under the microscope a clear base (perhaps not wholly devitrified) studded with numberless minute granules and rods of opacite, indicating a fluidal structure. The spots and bands prove to be groups of spherulites crowded together with irregular interlocking edges.* There is some green hornblende, generally in clustered granules, possibly associated with minute tourmaline. The larger felspar crystals resemble orthoclase or sanidine, and there is a little free quartz. It is therefore a rhyolite, but not one of the kind which, judging from this collection, is so common in Socotra. Of this [4203, 4213, 4214] are types, "compact red felstones or rhyolites," exhibiting on their weathered surfaces a beautifully distinct and rather parallel fluidal structure. This is most perfect in the second specimen. Microscopic examination shows it to be produced by ferrite-stained bands and lines of opacite granules and trichites with clearer interspaces. Small crystals of quartz, felspar, and little spherulites occur occasionally. The ground mass seems to be wholly devitrified. The first specimen shows traces of "flow brecciation" and is more unequally devitrified, with but few imperfect spherulites. The last [4214] has irregular spherulites and bands with a brush-like arrangement of microliths: it contains a little more free quartz. From "the banks of the Kereguiti stream, south of Hadibu," apparently intrusive in the green slaty rocks and the lower part of the overlying limestone, come pale buff or grey felsites. [4234] exhibits under the microscope an extremely minute devitrified structure, the slide is slightly clouded with ferrite and contains small scattered granules of quartz and felspar and minute specks of (?) hornblende. This rock is not unlike some of the most compact varieties of felstone from the Bala group of North Wales. From "the spurs of the Haggier range running towards the sea" come compact red quartz-felsites and rhyolites, generally resembling those already described. [4285] is cryptocrystalline, with a rather coraloid or arborescent structure, occasionally spherulitic; it is much stained with ferrite, contains a few scattered grains of quartz, decomposed felspar, iron peroxide, and hornblende or tourmaline, and is more probably from a dyke than from a flow.

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* As in the figures of a devitrified glass: Daubrée, 'Études Synthétiques de Géologie Expérimentale,' 1re Partie, pp. 170 and 171.

A pinkish quartz-felsite [4331] "from the interbanded group of rocks underlying the conglomeratic base of the limestone, south of Ma-aber," has a cryptocrystalline ground mass of quartz and decomposed ferrite-stained felspar; the former mineral, which is very abundant, has at first sight a rounded or polygonal outline, giving the rock a superficial resemblance to one of fragmental origin. In this ground mass are scattered grains of quartz, crystals of felspar, and (probably) magnetite. In the eastern part of the Haggier range near Adona the coarse red pegmatite is cut by dull purplish compact felstone dykes. [4399] has been taken from a vein only about ¾″ thick. The junction with the granite is beautifully exhibited in the slide, one or two small fragments of the latter being included in the former. This exhibits a cryptocrystalline structure, almost microcrystalline in parts, which, however, is in no respect remarkable. In the other [4368], also a junction specimen, the intrusive rock is minutely devitrified, shows slight fluidal structure, and has a general resemblance to the rhyolites already described; a small band of crushed granite occurs at the junction, as shown in the figure (Plate 7, fig. 3).

The felsites, which are associated with the argillites and calcareous rocks on the banks of the Kereguiti stream (Hadibu Plain), being apparently intrusive in them, are of a different character; they are compact flinty felsites of a pale buff or grey colour. One [4234] under the microscope exhibits an extremely minute cryptocrystalline structure, the ground mass being irregularly and lightly clouded with ferrite, containing specks of a hornblendic or chloritic mineral, with grains of quartz and felspar, some of which show the twinning of plagioclase.

Rhyolites of a purplish colour, exhibiting fluidal structure, and of the former type, occur among the boulders on the plain between Girgha and Omhari, and pinkish quartz-felsites come from the extreme north-west of the Girgha range, where, in the ascent of a valley from the Hadibu plain, "red granite, white granite, reddish felstone, and intrusive dark rock, alternate one with another with marvellous rapidity, and disappear under the limestone on the northern side." One of the quartz-felsites [4285] under the microscope exhibits a rather irregular cryptocrystalline structure, approaching coralloid or aborescent, here and there spherulitic; the spherulites having an irregular outline. The slide is stained with ferrite, contains a few scattered grains and imperfect crystals of quartz, decomposed felspar, with microliths of iron-peroxide, and hornblende or tourmaline. From the group of compact felsites or rhyolites, which are members of a very variable series of rocks near Gedidery, one of the most compact, of a purplish colour [4364], exhibits under the microscope a clear base, interspersed with minute aggregated granules of ferrite, varying from a sienna-brown colour to almost black, which gives a mottled aspect to the slide. Scattered in this are crystals of sanidine and plagioclase (? albite), and other crystals, now occupied by yellowish-brown secondary minerals and clotted ferrite, but which, judging from their outlines, have in some cases almost certainly been augite, others, however, may have been biotite. The felspar crystals have a curiously rounded outline, and contain numerous enclosures of brown glass. No free quartz is visible, so that the rock must be classed with the sanidine trachytes. The compact red felsites intrusive in granites "from the side of Khor, near the village of Gharrieh," are very much of the normal type, compact reddish or pinkish felsites, containing minute specks of quartz and of a greenish mineral, with small felspar crystals. [4359], examined microscopically, exhibits one of the slightly "arborescent" cryptocrystalline structures already noticed, and is clearly related to types described above. [4350], from the same region, is very closely related to the vein-granites in structure. The ground mass is microcrystalline; the quartz, which occurs in irregular grains of variable size, has many microlithic enclosures; there are one or two distinguishable crystals of orthoclase, and clusters of small crystals of biotite.

Several specimens of compact felsites or rhyolites have been brought from the district to the west of Kittim and then north-north-west as far as Azorah, where these, together with a conglomerate or agglomerate containing a similar rock, form a large part of the hills of this district. Ten slides have been examined microscopically, but after what has been already said it will be needless to do more than indicate their distinctive features. [4443] and [4482] are microcrystalline and exhibit an imperfect micrographic structure, the former having a more porphyritic structure. It contains a good quantity of plagioclase (? albite) among its felspars and a few specks of decomposed grown mica. [4458], [4472], and [4473] are reddish rhyolites with a cryptocrystalline ground mass, and more or less of a spherulic structure. The second is a particular interesting example, longish microliths of ferrite-stained felspar being scattered, or irregularly grouped, among clear quartz and occasionally formed into true spherulites, which are often enclosed by a dark ring. The microliths frequently indicate by their grouping an incipient micrographic structure (Plate 7, fig. 4). [4408] and [4413] are rhyolites showing a very marked fluidal structure, the former affording one of the best examples of "flow brecciation" that I have ever seen; fragments of lava, often differing considerably in their minor characteristics, being entangled in the matrix of the slide, showing that the partially solidified rock from time to time has been broken up and swept along by the pressure of the still liquid mass behind (Plate 7, fig. 5). The rock has been subsequently cracked and the fissures cemented by infiltrated minerals, among which is opal. The locality of the last two rocks is indicated as "close to Azorah (almost half way between the north and south coasts)." [4446] and [4448] are rocks of a dull purple colour, with small amygdaloids (Plate 7, fig. 6). The matrix consists of very minute elongated felspar microliths in a base, rendered almost opaque with ferrite and opacite; opal, chalcedony, and a clear mineral resembling felspar occupying the vesicles of the rock. Its relations are perhaps rather with the andesites than the true trachytes. [4431], from nearer Kittim, where it is associated with conglomerates and forms low hills, looking in places much like a lava flow, differs from the others in colour, being a compact pale grey rock like some of those which I have obtained from Moneadhmore Glen, in Arran, or even from North Wales; it has a rather variable cryptocrystalline base, ferrite-stained, in which are scattered many small felspar crystals (probably orthoclase), quartz, and a chloritic mineral. Apart from the confirmatory evidence given by Professor BALFOUR, I should have felt no hesitation in asserting that some, at least, of these rocks formed parts of lava flows; they are anterior in date to the limestone beneath which they pass.

From the hill of Afero, which, Professor BALFOUR states, reminded him of a volcanic neck, west-south-west of Azorah, approaching the eastern end of the Haggier range, come more reddish compact quartz-felsites and rhyolites of the usual type. A specimen [4436] which has been examined microscopically exhibits a fluidal structure, in parts cryptocrystalline, in parts rather microcrystalline, and much ferrite-stained; but it is needless to dwell on points of only varietal interest.

The last specimens from this group of rocks examined microscopically were collected *in situ* near Feraighey, on the southern side of the Haggier range. [4440] is a very compact pale Indian-red felsite, resembling a group described above [4458, &c.], but is a little more spherulitic. [4420] is nearer to [4446] in general character; there are some curious rounded concretionary spots of opacite, and perhaps one or two minute amygdaloids.

Mica-traps.

From Socotra we have one or two representatives of this outlying and rather vague but convenient group. One [4270] is from a dyke forming the floor of a gully on the northern side of the Haggier Mountains, inland from Hadibu. The gully is excavated in the coarse reddish pegmatite, which is cut by a vein granite [4264] (p. 282). This mica-trap is a rather compact variety consisting of a reddish felspar, mica, and a dull greenish mineral. Microscopically it consists of felspar, mainly orthoclase, in which are scattered numerous microliths of opacite, hornblende and mica, with larger crystals of brown and white mica, chiefly the former, hematite and hornblende, the last being often crowded with opacite and clear microliths, perhaps apatite. The rock then is a minette and follows its usual habit of occurring in a dyke or vein.

From the shore west of Hadibu, near the argillites (p. 289), comes a very minutely crystalline rock of dull-grey colour, which when microscopically examined consists of decomposed felspar, probably plagioclase, altered magnesia-mica, iron peroxide, a little hornblende, apatite (?), and interstitial quartz. This then is a kersantite.

From between Kami and the hamlet Ma-aber, on the Motaha, comes a minutely crystalline rock [4309] associated with felspathic granitoid rocks already mentioned (p. 276), which consists of pale grey felspar, mica, and some quartz, grains of each occurring porphyritically in the ground mass. The microscope shows that a good deal of the felspar is plagioclase, some being oligoclase, but orthoclase is also present; the larger crystals often enclose numerous clear microliths, there is also much biotite and some quartz, with grains of iron peroxide. The rock then is a quartz-kersantite.

Unaltered clastic rocks.

Passing now to the comparatively unaltered clastic rocks, we have first to notice the agglomerates or conglomerates from the district near Kittim, where they occur in large masses in association with the rhyolites as already described (p. 277). There are conglomerates and breccias (generally rather decomposed) of a compact red rhyolitic rock, some of which reminded me in appearance of the specimens of late Precambrian age, which are so common to the west-south-west of Bangor, a mile or two from the town.*

[4437] has been examined microscopically. It is composed of fragments of rhyolitic lava, subangular to rather rounded, with some quartz grains, containing very minute cavities, and a very little of a cementing matrix, which is generally stained with ferrite or viridite. The lava fragments vary considerably in their structures. One, containing some small felspar microlites, has a marked fluidal structure, being banded with very dark ferrite; it is almost certainly a fragment of scoria or from the slaggy exterior of a lava flow; others exhibit various fluidal and microlithic structures. Most of them are ferrite stained, and more or less devitrified: materials derived probably from more than one source. There are also some grains of viridite, these being probably altered fragments of augite or hornblende. The rock is a rhyolitic grit, the materials of which are certainly of volcanic origin, but probably (in this particular case) have been transported by water. The resemblance already noticed to the Welsh Precambrian rocks does not disappear under the microscope, and I think we have in this group of specimens evidence of volcanoes of very considerable geologic antiquity.

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* Quart Journ. Geol. Soc, vol xxxv, p. 309, and Geol. Mag., Dec. ii., vol. vii, p 298.

The conglomerates of rhyolite and felsite, which occur as boulders on the Nogad plain (p. 278), though their materials have been derived from the same group of igneous rocks, are probably far more modern than the last. These I have not deemed it necessary to examine microscopically. Breccias and conglomerates of various kinds are not infrequent in Socotra, but generally present no special interest. Some occur just at the base of the limestone series, others may be quite modern; subærial or stream detritus, cemented by tufaceous deposit.

Argillites.

This group of rocks, found underlying the limestone near Hadibu Bay (p. 276), is one upon the exact nature of which it is difficult to decide. I am not able to satisfy myself, from the specimens which I have received, whether there is a true cleavage or not; the rock has a somewhat porcellaneous aspect, as if it were at least of palæozoic age, but as there appear to be intrusive rocks in the neighbourhood, one hesitates without further evidence to assign it to a remote geologic period. As, however, there is at the base of the limestone a breccia of fragments, seemingly in the same mineral condition, and as there seems to be an unconformity here, the interval between the two groups of rocks is probably considerable. Shaly bands are, indeed, said to occur in the lower part of the limestone, but it is possible that these may be formed of detrital material from the older group. Microscopic examination does not throw much light on the history of the latter. The specimens are composed of earthy material and minute, rather angular, fragments of a clear mineral (quartz, with possibly some felspar), and a number of exceedingly minute, doubly-refracting, rather highly-tinted granules; these are probably in some cases a variety of hornblende, in others epidote.

Limestone and dolomite.

Several specimens of the massive limestones which form so much of the upland district in Socotra are among Professor BALFOUR'S collection. Generally they are compact in texture, with a clean subconchoidal fracture, buff, whitish, or sometimes pinkish in colour; in aspect often remarkably like the ordinary dolomitic rock of the Italian Tyrol. The weathered surfaces are sometimes curiously rough and cavernous. I have examined microscopically a few specimens from different parts of the island. The first [4168] is one of a series collected in the district extending from the head of the Gollonsir Valley towards the Kadhab Plain. These are compact in structure with a subconchoidal fracture, buff coloured and pinkish, the former being more crystalline. The specimen examined is a very pretty pinkish rock that would take a good polish. It contains numerous organisms, both tolerably perfect and fragmentary, among which foraminifera abound. Among these we can recognise* *Globigerina* (abundant) *Rotalinæ*—probably two forms of *Planorbulina*, one like *Ammonoides*, and the other outspread like *Mediterranensis*—*Textularia* (more than one form), *Amphistegina*, and possibly a small *Nummulites*, with fragments of mollusca, polyzoa (?), and of other organisms which I cannot recognise. The matrix is rather earthy.

A specimen [4048] from the cliff capping the ridge above Hainjou (west end of the island) is a very compact pale cream-coloured limestone. Microscopic examination shows a matrix of dusky grey material, with minutely-crystallized calcite and probably a little dolomite, which occurs also in thin veins, and some minute granules which may be glauconite. It contains several *Rotalinæ*, possibly *Planorbulina* and *Discorbina*, and perhaps *Rotalia*, together with a large fragment rather resembling a piece of a *Cristellaria*.

Buff limestones from the Kadhab Plain, [4188] and [4195]. The former contains numerous mineral fragments, apparently quartz and felspar, with organisms broken and whole, among which *Amphistegina* and *Miliola* are recognisable, and bits of the shells of mollusca. The organisms are more or less dolomitized. The latter contains many *Globigerinæ* with *Amphistegina* (small), perhaps an *Orbitoides*, and fragments of mollusca and polyzoa. This rock also is slightly dolomitized.

A specimen of limestone [4107] from the plateau west of Ras Mom, near the eastern extremity of Socotra, is a buff-coloured rock with a minute oolitic structure, the weathered surface being full of small holes. Under the microscope we found a clear dolomitic matrix full of darkish rings, circular to irregular oval in form, with a radial-fibrous structure, and as a nucleus commonly either a fragment of quartz or of a foraminifer, though sometimes they inclose simply the matrix; sometimes also the concentric structure extends to the centre of the oolitic grain.

[4456]—a very compact pale buff limestone from the top of Ras Mom, in appearance almost exactly like a bit of Schlern or Dachstein dolomite, is shown on microscopic examination to be a true dolomite. It contains numerous organisms, which, as is commonly the case, are more completely dolomitized than the matrix, and so rendered rather difficult of identification. There are many foraminifera, among them certainly *Amphistegina*, and perhaps *Nummulites*, *Globigerina*, with fragments which I believe to be polyzoa.

As regards the geologic age of these limestones, *Globegerina* has had a long existence, like *Textularia*; *Nummulites* began long since, and still lingers, but was most abundant in the eocene time; *Orbitoides* began in cretaceous and disappeared before the latest part of the tertiary age; *Amphistegina*, however, which is one of the most abundant foraminifers in these rocks, ranges from the miocene upwards, hence it is probable that the age of these Socotra limestones is miocene.

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* As I have not worked much at thin sections of foraminiferal limestone, I submitted all these slides to Professor RUPERT JONES, F.R.S. With his wonted kindness, though at the time far from well, he examined them and gave me a series of notes, which are embodied in the text, so that the identifications rest upon his authority, than which it would be difficult to find a better.

Conclusion.

From the above remarks it would appear that in Socotra we have, as the foundation stones and "core" of the island, if the phrase be permissible, a mass of rock of very great antiquity. There is clear evidence of the presence of gneissic rocks which in their lithological characters resemble closely those which exist in the north-west of Scotland, the Malvern Hills and one or two other localities in our own island, in Northern America, and in many other parts of the globe. By whatever name these may be called, and however they may be correlated one with another, it is evident that their antiquity is enormously great, and that they had attained their present mineral condition before the earliest palæozoic rocks were deposited. Associated with these are granites, which, though of later date, are probably also of great antiquity. In the Sinai peninsula we have also gneiss, schists, and various granitic rocks. Of the latter I possess a small collection, given to me by Professor E. H. PALMER,[*] and the resemblance of some of the specimens to those of Socotra is very remarkable. In the geological notes added by the Rev. F. W. HOLLAND to the Report of the Ordnance Survey (ch. viii.) we find it stated, indeed, that the prevailing rock, in the Sinai region, is syenite (by which term probably hornblendic granite is meant), so that out of several hundred specimens he only possessed two or three of true granite. This may be, but my specimens from the summits of Serbal, of Jebel Musa, and Um Shomer closely resemble some of the Socotra rocks, especially the first and second, which are coarse reddish granites composed almost wholly of quartz and felspar (pegmatites). From the summit of Serbal also comes a finer grained granite, and I have an ordinary granite from Wady Sigillia. Other specimens, exactly as in the Socotra collection, might be either true igneous or highly metamorphic rock. In the Sinai region the old gneiss appears only to have been recognised in the northern part, where it forms an irregular trough to the north of Jebel Serbal, the higher peaks (like the Haggier range in Socotra) being granite. I miss, however, from the Socotra district, or find but feebly represented—for there is one specimen which may belong to it—the friable variably-coloured sandstones which form so marked a feature in some parts of the Sinaitic peninsula (*e.g.*, Wady Mokatteb)[†]. These, after having been assigned to more than one geological epoch, were referred by Mr. HOLLAND, on the evidence of fossils, to the carboniferous. In Sinai, as in Socotra, we have huge masses of limestone, which in like way form great plateaux—as, for example, the Tih—and were deposited in an ocean, in which the well-known peaks of Sinai probably formed rocky islands, but deposition there commenced at an earlier period than we can venture with the evidence at present before us to claim for Socotra, for the limestones of Sinai are assigned to part of the cretaceous and to the eocene age; the Nummulitic limestone, for example, being finely developed as in Wady Dhaghadeh. Coraliferous beds of miocene age are, however, found in that region.* The rocks of Sinai are cut by dykes of "basalt, greenstone, and porphyrite," the first of which, as in Socotra, are probably comparatively modern, but we do not find there, so far as I can learn, representatives of the great group of the quartz-felsites and rhyolites which seem so enormously developed in Socotra and were certainly connected with active volcanos. The geological age of these cannot be determined. They are undoubtedly older than the limestone group; so that, if no part of this is earlier than the middle tertiary, they might belong to any geological period between that and the latest Precambrian, to the volcanic rocks of which they have, indeed, considerable resemblance. I am not aware that the "argillite" of Socotra—of which I can only say that it is older than the limestone—is represented in Sinai. As here, so also in Socotra, there are basalts of comparatively late geologic age—post miocene—and in the latter some compact trachytic rocks, which, however, differ from the older rhyolites, and are generally paler in colour.

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[*] He was a member of the Sinai Survey Expedition in 1868–9. Two months after this paper was read, he was murdered by Arabs. The circumstances of his death are well known, but I may be allowed to pay a passing tribute to the memory of a singularly accomplished and learned man and very valued friend.

[†] This absence of the sandstone is probably an accident (the rock being very friable), for Professor BALFOUR mentions "purple sandstones" (see p. 275).

We have, then, in Socotra, as it seems to me, evidence of rocks of an immense, and a land surface of a very great, antiquity. Excepting this argillite of uncertain age and limited extent, and perhaps some sandstone (also local), there is no evidence in the specimens before me to show that this island was submerged during any part of the palæozoic or mesozoic period.† During the kainozoic it undoubtedly shared in the downward movement which affected so large a portion of the globe in and about the North African and mid-Asiatic districts; but I should infer that the invasion of the sea commenced much earlier in the Sinaitic peninsula, and think it possible that the topmost peaks of the Haggier Mountains were at no time wholly submerged. As it again rose from the waves—perhaps being for a while connected with the African continent—the meteoric forces resumed their work of sculpture and the waves began their work of insulation. Since then the fauna and flora have undergone their own modifications, but in the Haggier Hills we have probably a fragment of a continental area of great antiquity and of a land surface which may have been an "ark of refuge" to a terrestrial fauna and flora from one of the very earliest periods of this world's history.

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* BAUERMAN, Quart. Jour. Geol. Soc., vol. xxv., p. 37.

† "Africa south of the Sahara was probably a stable area during many of the alterations of the relative levels of land and sea of the north and of Europe"—Professor P. M. DUNCAN, Presidential Address to Geol. Soc. 1877, Journal, vol. xxxiii., p. 86 (Proceedings). West of the Sinai Peninsula old schists and granite crop out in Egypt, and east of it on the opposite side of the Gulf of Akabah, flanked in both cases by "Nubian sandstone." See the map attached to the Presidential Address to the Geologists' Association (delivered November 3, 1882), by Mr. W. H. HUDLESTON (Proceedings, vol. viii., p. 1), in which is given an admirable summary of the Geology of Palestine and the neighbouring districts.

DESCRIPTION OF PLATES.

PLATE 6.

Sketch map of Socotra.

The information at the Author's disposal has not enabled him to attempt to lay down the geological boundaries with any accuracy or to insert minor features, such as dykes, &c. It will be understood that the rock in the parts left blank (except where alluvial plains occur) is limestone of approximately middle tertiary age. Only the more important names are placed on the map.

PLATE 7.

1. Labrador diorite from base of Azalin (p. 281) × 20.
   - (*a.*) Labrador felspar, in parts rather decomposed; the striping is indicated as when a polarizing apparatus is used. This has consolidated before the hornblende.
   - (*c.*) The sap-green hornblende, within which is a crystal of iron oxide, probably ilmenite, and the pyroxenic constituent (*b*) of the text. Since writing this I have noticed that one of the two vertical sections of the mineral extinguishes when the cleavage lines coincide with the vibration planes, and is rather dichroic, giving reddish and greenish tints, according as the vibration of the polarized beam is perpendicular to or parallel with the cleavage lines of the mineral. The cross sections, though much resembling augite, are rather abnormal in some respects. Thus, though the evidence is insufficient for certainty, I think it possible that we have here the rhombic pyroxene (hypersthene ?) lately described by Mr. WHITMAN CROSS ('Amer. Jour. Science,' Feb., 1883) and Mr. TEALL ('Geol. Mag.,' Dec. ii., vol. x., p. 145).
2. Vein granite from the slopes above Hadibu (p. 283) × 20.

   The figure contains two crystals, selected from different parts of the slide, to represent those crystals which while resembling hornblende in cleavage have optical characteristics agreeing with tourmaline. They are extremely dichroic, changing from a clear sap-green to a blue-black.

3. Junction of felsite and granite. Haggier range, near Adona (p. 285) × 20.
   - (*a.*) Very compact felsite.
   - (*b.*) Crushed granite.
   - (*c.*) Granite.

   Drawn with polarized light.
4. Rhyolite with spherulites. Azorah district (p. 287) × 40.

   Selected as a fairly typical example of a rather frequent type of rock.
5. Rhyolite exhibiting "flow brecciation," near Azorah (p. 287) × 20.

   The dark streaky part (*a*) is the general ground mass of the rock in which fragments are entangled; (*b*) exhibits small lath-like crystals of a decomposed felspar in a ferrite stained ground mass somewhat resembling that of the last slide; (*c*) exhibits a minute "speckled" devitrification structure; (*d*) with a high power exhibits minute felspar microlites crowding a dark ferrite-stained base, while in the centre of the slide is a fragment showing a well-marked fluidal structure, the lighter parts of which are devitrified. It is of course possible that some of the fragments may be entangled lapilli, but the absence of a slaggy border and the general cleanness of their edges corresponds better with the idea of their having formed parts of the flow.
6. "Trachyte" with minute amygdaloidal cavity. Near Azorah (p. 287) × 30.

   The minute felspar microliths are slightly more evident in the slide than in the drawing, and one or two of the cavities are larger than that figured, but unfortunately their contents have partly torn away in grinding.

VII. *Experiments, by the Method of* LORENTZ, *for the further Determination of the Absolute Value of the British Association Unit of Resistance, with an Appendix on the Determination of the pitch of a Standard Tuning-Fork.*

By Lord RAYLEIGH, *F.R.S., Cavendish Professor of Experimental Physics in the University of Cambridge, and Mrs.* H. SIDGWICK.

Received December 8, 1882,—Read January 11, 1883.

§ 1. IN this method, which was employed by LORENTZ in 1873,[*] a circular disc of metal is maintained in rotation at a uniform and known rate about an axis passing through its centre, and is placed in the magnetic field due to a battery current which circulates through a coaxal coil of many turns. The revolving disc is touched at its centre and circumference by two wires. If the circuit were simply closed through a galvanometer, the instrument would indicate the current due to the electromotive force of induction acting against the resistance of the circuit. The electromotive force corresponding to each revolution is the same as would be generated in a single turn of wire coincident with the circumference of the disc by the formation or cessation of the battery current. If this be called γ, and M be the coefficient of induction between the coil and the circumference, m the number of revolutions per second, the electromotive force is $m\mathrm{M}\gamma$. In the actual arrangement, however, the circuit is not simply closed, but its terminals are connected with the extremities of a resistance R, traversed by the battery current, and the variable quantities are so adjusted that the electromotive force $\mathrm{R}\gamma$ exactly balances that of induction. When the galvanometer indicates no current, the following relation, independent, it will be observed, of the magnitude of the battery current, must be satisfied—

$$\mathrm{R}=m\mathrm{M};$$

and from this, M being known from the data of construction, the absolute resistance R of the conductor is determined.

One of the principal difficulties to be overcome arises from the smallness of the resistance R, necessary for a balance, even when m and M are both increased as far as possible. LORENTZ employed three resistances, ranging from ·0008 to ·002 of a mercury unit, and he evaded the necessity of comparing these small resistances with ordinary standards by constructing them of actual columns of mercury. His result was accordingly obtained directly in terms of mercury, and was to the effect that

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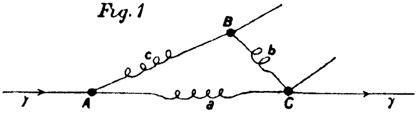
[*] POGG. Ann., cxlix., p. 251.

$$1 \text{ mercury unit} = \cdot 9337 \times 10^9 \text{ C.G.S.}$$

differing nearly 1 per cent. from the value (·941) obtained by ourselves.

§ 2. Under the conviction that this method offers in some respects important advantages, and influenced also by the fact that the arrangements for producing and measuring the uniform rotation necessary were ready to our hands, we determined to give it a trial, in the hope of obtaining confirmation of the results already arrived at by ourselves and by GLAZEBROOK with other methods. At first the intention was to follow LORENTZ in using for the resistance a glass tube full of mercury, with two points of which contact would be made by platinum wires passing through the glass. It appeared, however, that there would be difficulty in making the measurements with the degree of accuracy aimed at. If the wires were sealed into the glass, the section would probably be rendered irregular. An attempt was made to avoid this difficulty by using a tube from which the ends had been cut with the aid of heat. After small nicks had been filed sufficiently deep to receive the platinum wires, the ends were replaced in their original positions and secured with shellac. In this way a satisfactory uniformity of section near the points of derivation could be attained, but the measurement of the distance between these points, which is required to be known with full accuracy, was rendered difficult by the presence of the cement. It is possible that these difficulties might have been overcome, but at this point a method of shunting occurred to us, allowing the use of mercury to be dispensed with. Merely for the purpose of connecting the mercury unit with the B.A. unit or other standard of resistance, it would not be desirable to use tubes of such large bore.[*] This problem may more conveniently be taken by itself, and has already been treated by us in a former communication to the Society.[†]

Fig. 1



§ 3. In the shunt method the greater part of the main current γ passes on one side through a relatively small resistance a (see fig. 1), and the difference of potentials at the points of derivation B, C, is due to the passage of a small fraction only of the total current, the resistance $(b+c)$ being great compared with a. If at the same time b be small relatively to c, the difference of potentials is doubly attenuated. Its value for a given main current γ is found at once from the consideration that the current divides itself between the two branches in the inverse ratio of the resistances. The current through b is thus $\frac{a}{a+b+c}\gamma$, and the difference of potentials at the points of derivation is $\frac{ab}{a+b+c}\gamma$. The quantity $\frac{ab}{a+b+c}$ thus takes the place of R in the simple formula, and is called the effective resistance. By taking for instance $a=\frac{1}{2}$, $b=1$, $c=100$, we get an effective resistance of about $\frac{1}{200}$; and the resistances employed may be those of ordinary resistance coils, capable of accurate comparison with the standards.

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[*] If the distance between the points of derivation were 1 mètre, $R = \cdot 002$ mercury unit would require a section equal to 500 square millims.

[†] Phil. Trans., 1883, p. 173.

§ 4. In designing the apparatus we were influenced by the fact that we had at our disposal two very suitable coils of large radius, wound some years ago by Professor CHRYSTAL, the same in fact as were used by Mr. GLAZEBROOK in his investigation by another method. By bringing the two coils close to one another and to the plane of the disc, the inductive effect is rendered a maximum. This arrangement accordingly was the one first experimented with, as being the most likely to prove successful.

The diameter of the disc is limited by two considerations. If it be too small, the whole inductive effect, and with it the sensitiveness of the arrangement, suffers. On the other hand if it be too large, the circumference enters the more intense region of magnetic force which lies near the wire, and the coefficient of induction changes its value rapidly when any alteration occurs in the mean radius of the coils, or in the diameter of the disc, and thus the final result becomes too sensitive to errors in the magnitudes of these elements. In the Phil. Mag. for November, 1882, the reader will find a calculation of the values of M for various cases, and a general comparison of the principal methods for determining absolute resistance especially in respect of errors arising in connexion with the fundamental linear measurements. For the experiments now to be described, the diameter of the disc was chosen so as to be somewhat more than half that of the coils (§§ 22, 23).

§ 5. The disc was of brass and turned upon a solid brass rod as axle. This axle was mounted vertically in the same frame that carried the revolving coil in the experiments described in a former communication to the Society* (see Plate 48), an arrangement both economical and convenient, as it allowed the apparatus then employed for driving the disc and for observing the speed to remain almost undisturbed. The coils were supported horizontally upon wooden pieces screwed on the inner side of the three uprights of the frame.

During the earlier trials, extending over the month of May, 1882, the edge of the disc was bevelled, and contact was made with it by means of a brush of fine copper wires held in a nearly vertical position. No sufficiently regular results could be obtained until the sliding surfaces were amalgamated, and even then there were discrepancies between the work of one day and that of another, whose cause was not discovered until a later period. It soon became manifest, however, that the bevelled edge would not answer the purpose, for it cut its way by degrees into the wires of the brush in such a manner as to render the effective radius uncertain. The substitution of a cylindrical for a bevelled edge promised better results. The width of the edge

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* Phil. Trans., Part II., 1882.

(equal to the thickness of the disc) was $4\frac{1}{2}$ millims. and allowed sufficient room for the contact of the brush though placed tangentially. In this way broader bearing surfaces were available, and the small extension of the contact in the direction of the axis is unobjectionable, provided everything is arranged symmetrically with respect to the middle plane of the disc.

As will presently appear, the success of the method is independent of any constant thermo-electric force at the sliding contact, but it is evident that good readings cannot be taken if the thermo-electric force changes its magnitude often and suddenly. It was found advisable to renew the amalgamation of the edge at the commencement of each day's work. The excess of mercury, if any, attaches itself to the brush, and does not appear to render the diameter of the disc uncertain.

The inner contact was made in a similar manner by a brush pressing against the shaft itself at a place a little below that at which the disc was attached. The coefficient of induction to be employed in the calculation is the difference between the coefficients for the coil and the outer and inner circles of sliding contact respectively, but the latter is quite subordinate (§ 25).

§ 6. The disc was driven by the same water-engine that was employed for the revolving coil of former determinations,* the connexion being made by a long cord passing round a wooden pulley attached to the lower part of the shaft. To the upper face of the disc was cemented a circle of paper on which were marked a series of circles of alternately black and white teeth. One observer looking through the prongs of an electro-magnetically maintained fork regulated the speed of the disc by application of the necessary friction to the driving-cord, which passed through his fingers. When one of the series of circles is seen to be stationary, a simple and easily expressed relation is established between the frequency of the fork and that of revolution. At intervals the number of beats per minute is counted between the notes of a standard fork, and (the octave of) the electric fork. There is no difficulty in thus determining the speed of rotation to within one part in 10,000. With respect to the absolute pitch of the standard fork itself, see the Appendix to this Memoir.

§ 7. When the disc is caused to rotate, and the galvanometer circuit is closed, a deflexion is observed, although the battery which generates the main current is not in action. This deflexion is due to two causes—thermo-electric force at the sliding contact, and induction dependent upon the vertical component of the earth's magnetism. Although not a direct source of error, this deflexion is better avoided, both for convenience in reading the galvanometer and because it implies the actual passage of a not insensible current through the sliding contacts and thus brings into consideration the *resistance* of these contacts. The compensation was effected by the introduction of an opposing electromotive force; for which purpose two terminals of the galvanometer circuit J, K, fig. 2, instead of being connected directly, were attached by binding screws to two points on a stout copper wire forming part of a circuit which included a sawdust DANIELL (L) and a resistance coil of 100 ohms (M). By shifting one of the binding screws, the galvanometer reading, in the absence of the main battery current, and after attainment of the proper speed, was made to be nearly the same as when the galvanometer contact was broken.

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* Proc. Roy. Soc., May 5, 1881, p. 112; Phil. Trans., Part II., 1882.

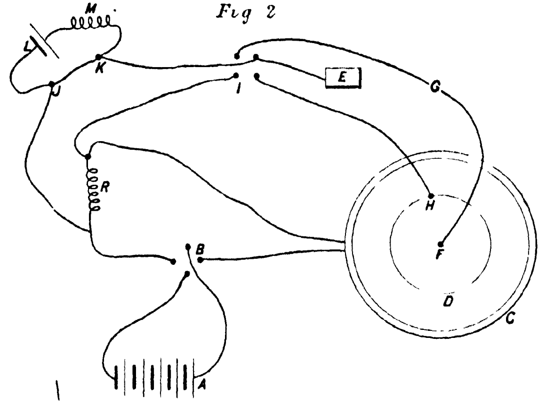
Fig 2

§ 8. The general plan of the connexions and the *modus operandi* will now be intelligible from fig. 2. The poles of the battery A, consisting of 20 DANIELL cells, were connected with a mercury reversing key B, the two positions of which were distinguished by the letters E and W (east and west). From thence the current passed through the induction coils C and the equivalent resistance R, of which the details are reserved for the moment. The reflecting galvanometer, G, is placed at a considerable distance in order to avoid the direct influence of the coils, and is connected with the inner sliding contact, F. Its resistance is about ½ ohm; and by the aid of the compensating magnet the vibrations of the needle were made slow enough to be readily observed. The terminals of the galvanometer branch, which includes also a commutator, I, are connected to the extremities of the resistance, R.

If, while the disc is maintained in uniform rotation, the reading of the galvanometer is the same whichever way the battery key may stand (correction being made, if necessary, for a direct effect upon the needle), it is a proof the contemplated balance is actually attained. In this way all disturbance from the earth's magnetism, and from thermo-electric forces whether situated at the sliding contacts, or within the resistance coils of which R is composed, or at any other part of the galvanometer circuit, is eliminated from the result. The adjustment is effected by varying a comparatively large resistance, taken from a box, and placed in multiple arc with one of the components of R.

§ 9. In actual work, however, it is not necessary, or even desirable, to hit off the balance with great accuracy. An unmistakable difference of readings when the battery key is put over, is rather an advantage than otherwise, as giving an indication that the circuits are properly closed. The plan adopted was to take a series of readings of the effect (E—W) of reversing the battery current with an effective resistance R_1, not very different from R. Single readings were liable to considerable irregularity in consequence of change in the friction at the sliding contacts, and of momentary variations in the speed. These errors cannot possibly be systematic, and are in great measure eliminated in the mean of a series. Having thus obtained the difference of galvanometer readings (E—W) corresponding to R_1, we altered the resistance in multiple arc so as to change R_1 into R_2, the difference being some such fraction as $\frac{1}{100}$ of the whole, and in such a direction that the sign of E—W is changed. The two series give by simple interpolation (after correction for the direct effect) the true value of R, that is the effective resistance corresponding to the balance. In order to get the best result relatively to the time occupied, the number of observations of E—W in each set was taken roughly in inverse proportion to the values. To diminish the influence of a progressive change in the strength of the battery current, the observations with R_2 were interspersed between those with R_1 as effective resistance. The readings were usually taken continuously, with no more delay than was necessary to allow the vibrations of the needle to become of moderate extent after each change. When they were completed, the driving cord was reversed, as well as the commutator, I, and a similar set of observations was taken with rotation in the opposite direction.

§ 10. In the earlier experiments the resistance coils composing the effective resistance were arranged as in fig. 1, in which A, B, C may be supposed to represent mercury cups, the bottoms of which were formed of amalgamated copper discs. On these discs rested the amalgamated terminals of the various resistance coils and connecting wires. The shunt *a* consisted of two unit coils in multiple arc, between which the greater part of the main current was equally divided. The magnitude of the main current was less than $\frac{1}{10}$ ampère. The resistance *b* between the points of derivation was a unit, while the third resistance *c* was alternately 105 and 106.

In reckoning the resistance of the galvanometer circuit we have to include *b*. The remainder scarcely exceeds the $\frac{1}{2}$ ohm due to the galvanometer itself. It appears therefore that the deflections obtained with the arrangement described are only one-third part as great as they would be if a quite small resistance were substituted for the unit in *b*. As the sensitiveness appeared likely to be inadequate, we afterwards replaced the unit by $\frac{1}{10}$, using for *c* a coil of ten units. As in this case the addition or subtraction of a whole ohm in *c* would make too great a difference, the adjustment was obtained by varying a comparatively large resistance placed in multiple arc with *a*.

In the light of subsequent experience it is doubtful whether this change was an improvement. The increase of galvanometer deflection was not really of much advantage, since the difficulty of getting sharp results arose from electromotive disturbances, and these were magnified in the same proportion. It would probably have been better to have retained the unit in *b*, and to have replaced the galvanometer by one of higher resistance.

§ 11. Preliminary trials having given apparently satisfactory results, we proceeded to make regular series of observations in the manner already described. We had not gone far before anomalies revealed themselves of such a character as to prove that we were not yet masters of the method. It usually happened that each day's observations agreed well together, showing that the sensitiveness was sufficient; but when we came to compare the results obtained on different days unaccountable discrepancies became apparent. The first result of the more severe criticism to which the arrangements were then subjected was to show that sufficient thought had not been given to the question of insulation. The wire composing the induction coils, or rather one extremity of it, is necessarily at a high potential, and a very moderate leakage from the coils to the frame, and thence to the disc, might cause great disturbance. Some such leakage was in fact detected on application of appropriate tests. Ebonite insulation was accordingly introduced into the supports of the coils. The battery was carefully insulated from the ground, as was also the frame carrying the revolving disc, and other precautions were taken which it is unnecessary here to detail. For the sake of definiteness one point of the galvanometer commutator was connected to earth. With these improvements tests were satisfied more severe than that of actual use, and these tests were renewed at intervals during the spinnings.

The results however still showed that some defect existed which we had not yet succeeded in detecting. It made no appreciable difference which way the disc rotated, but the means of different days' work failed to exhibit the desired accordance. Two months' work had already been spent upon the experiments, and we had begun to despair of a satisfactory issue, when it occurred to us that the connexion of the coils for compounding the effective resistance was faulty.

§ 12. By reference to fig. 1 it will be seen that the main current traverses part of the cup C, and that part of the same cup is also included in *b*. Now, although for all ordinary purposes the resistance of the parts of the cup might be neglected, in the present case it is the small effective resistance R with which it comes into comparison. If we aim at an accuracy of $\frac{1}{10000}$, we cannot afford to overlook a resistance entering in this manner, even though it may not exceed $\frac{1}{2000000}$ ohm. The discrepancies were doubtless due to small differences in the position of the wires and coils in cup C, moved as they were from day to day in order to verify the soundness of the contacts.

In order to avoid the difficulty we have only to take care that no part of *b* can possibly be traversed by the main current, and this is easily done by the introduction of another mercury cup. Fig. 3 shows the arrangement adopted. The main current enters at the cups A and D, and the greater part is taken by the two unit coils in multiple arc whose terminals rest in these cups. The galvanometer terminals are led into two other cups B and C. The ends of these are beaten flat and the legs of the $\frac{1}{10}$ rest upon them. The connexion between C and D was through a stout copper rod, which may be regarded as part of c. For the first series the connexion between A and B was through a single coil of 10 units' resistance, replaced in subsequent series by other coils giving altogether 16 and 20 units' resistance respectively.

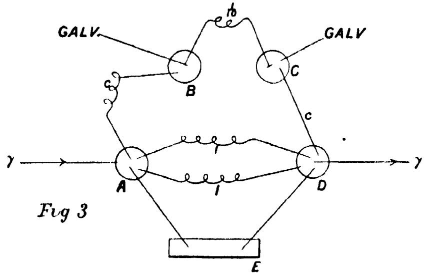

Fig 3

To make the necessary adjustment and variation of resistance, a box, E, was placed in multiple arc with the two unit coils. The resistances taken from the box were afterwards carefully determined, but they enter into the final results in quite a subordinate manner.

§ 13. Further trials now led to the satisfactory conclusion that the defect was remedied, for the means obtained on different days agreed well together, even although the resistance coils were taken down and remounted in the interval. As we had now every reason to suppose that our experiments would have a successful issue, we proceeded to make the final adjustments preparatory to a complete series of observations.

In the first and second series the two coils were near one another, separated only by three slips of glass, and held firmly together by wooden clamps. The adjustments presented no particular difficulty. By means of an iron finger clamped to the disc and carried gradually round, it could be verified that the coils and disc were concentric and in parallel planes. The coils were gradually wedged into their places, and secured when their mean planes occupied the desired symmetrical positions relatively to the disc. It is evident that errors of maladjustment influence the result only in the second order.

§ 14. Experience in this series having shown that the arrangement was satisfactory, and that the sensitiveness was fully sufficient, we proceeded to make a second series of observations without displacement of the induction coils, but at a speed of rotation lower than before in about the ratio of 16 : 10. This, of course, entailed a corresponding change in R, which was effected by increasing the component c. An agreement between the final results of the two series would give an important confirmation, inasmuch as leakage of electricity from the main circuit into the galvanometer branch would exert a different influence in the two cases. The observations were not reduced until some time afterwards, and it then appeared that the agreement was even better than it would have been reasonable to expect.

§ 15. The final number, $\cdot 9867 \times 10^9$, expressing the value of the B.A. unit in absolute measure as determined by these two series of observations, is almost identical with that previously obtained by ourselves, and by GLAZEBROOK, using other methods. With respect to the independence of these determinations, the only thing calling for notice is the fact that the same induction coils were employed both by GLAZEBROOK and in the present investigation. In other respects there has been, we believe, scarcely any point of contact. But it is evident that an error in the measurements of mean radius of these coils must propagate itself into both results. The point to which we now wish to direct attention, is that the error of mean radius will influence the final number in *opposite directions*. In the method employed by GLAZEBROOK, an under-estimate of the mean radius would lead to an under-estimate of the induction coefficient, whereas with us it would lead to an over-estimate of that quantity. So far, therefore, as the error of mean radius is concerned, it would appear that the use of the same coils is far from impairing the value of the results. Even with respect to the number of turns, an error, if that be supposed possible, would affect the results in a different manner, for GLAZEBROOK was concerned with the *product* of the numbers for the two coils, while we evidently are concerned with the *sum*.

§ 16. In researches of this kind it is proper to calculate the influence upon the result of errors in the fundamental measurements. The value of M depends upon three linear quantities: the radius of the disc (a), the mean radius of the two coils (A), and the distance between their mean planes ($2b$). In the present case, however, the latter element enters in a very subordinate degree. From § 25 it appears that

$$\frac{dM}{M} = -1\cdot 4 \frac{dA}{A} + 2\cdot 4 \frac{da}{a}.$$

It has been shown* that these conditions compare favourably with those of most of the other methods that have been employed. From its nature a is much more easily measured than the diameter of a coil.

§ 17. The results deduced from the several days observations, when corrected for slight variations of temperature of the resistance coils, &c., exhibit a remarkable accordance. By reference to the tables (§ 27) the reader will see that the maximum divergence from the mean in Series I. is only about one part in 4000, while in Series II. it is even less. We were thus encouraged to carry out a modification of the method which we had had in view all along, and the results of which would be in great measure independent of those of Series I. and II.

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* Phil. Mag., Nov., 1882.

§ 18. The modification referred to relates to the position of the induction coils relatively to the disc. In the arrangement with which we have been dealing hitherto, the mean planes of the coils are nearly coincident with that of the disc, and the accuracy of the final number depends upon an exact knowledge of the mean radius of the coils. It has, on the other hand, the advantages of being practically independent of measurements parallel to the axis, and of giving the maximum coefficient of induction. In the new arrangement the coils are separated to such a distance that the *result is nearly independent of a knowledge of the mean radius.* How this may come about will be readily understood by considering the dependence of the coefficient of induction M upon A, when a and b are given. It is clear that M vanishes, both when A is very small, and also when it is very large; from which it follows that there must be some value of A for which the effect is a maximum, and therefore independent of small variations of A.

In carrying out this idea, it is not necessary to approach the above defined state of things very closely; for of course we have in reality a good approximate knowledge of the value of A. In our apparatus the distance of mean planes was about 30 centims., so that $b =$ about 15 centims. ($A=26$, $a=16$). From the calculations in § 25 it appears that with the actual proportions

$$\frac{dM}{M} = +\cdot 12\,\frac{dA}{A} - \cdot 96\,\frac{db}{b} + 1\cdot 8\,\frac{da}{a};$$

so that the error of A enters in quite a subordinate degree. The positive coefficient of dA shows that with the given coils and the given disc the separation was somewhat too great to secure the utmost independence of dA.

§ 19. The success of this arrangement depends principally upon the degree of accuracy with which b can be determined. The two rings upon which the coils are wound were held apart by three equal distance-pieces, against which they were firmly pressed by wooden clamps. The distance-pieces were hollow, of massive brass, and the terminal faces were carefully turned. Central marks upon them facilitated the adjustment of the coils into the symmetrical positions. The distance of mean planes does not however depend solely upon the distance-pieces. Even if we could assume that the mean planes are symmetrically situated relatively to the grooves in which the wire is wound, we should still have to take account of the thicknesses of the flanges. All uncertainty in this matter is eliminated by following the plan adopted by GLAZEBROOK of reversing the rings (without interchange), and then repeating the measurements. Whatever may be the situation of the mean planes and the thicknesses of the flanges, the mean result thus obtained corresponds to a distance equal to the length of the pieces *plus* half the total outside thicknesses of the rings. These quantities can all be measured with great precision, and as easily after the coils are wound as before. Full particulars are given in § 24. There can hardly be a doubt but that the determination is much more accurate than that of the mean radius of a coil; and, what is also of some importance, it admits of repetition at pleasure with comparatively little trouble.

§ 20. The sensitiveness of this arrangement was about the same as in Series II., and the table shows a good agreement among the results obtained on different days. The final number from this series is ·9868, almost the same as from Series I. and II.

The small difference of effective resistances required for balance in the two positions of the induction coils, amounting to about one part per thousand, is almost exactly accounted for by the small difference of distances of mean planes in the two cases, as deduced from Professor CHRYSTAL'S measurements of the thicknesses of the flanges. In the first position (see § 24) the coils are nearer together by almost exactly one part per thousand, a difference which, according to the formula given above (§ 18), should be reproduced almost without change in M and therefore in R, the greater values of M and R corresponding to the smaller distance.

§ 21. If we combine all the results of the present investigation, giving equal weights to the two arrangements of the induction coils, we have

$$1 \text{ B.A. unit} = \cdot 98677 \times 10^9 \text{ C.G.S.}$$

With use of the ratio between the mercury unit and the B.A. unit found by us (Proc. Roy. Soc., May, 1882), this gives

$$1 \text{ mercury unit} = \cdot 94150 \times 10^9 \text{ C.G.S.};$$

or, which is the same thing, the ohm is the resistance of a column of mercury at 0° centigrade whose section is 1 square millim., and whose length is

1062·14 millims.

We now pass on to the details of the measurements.

DETAILS OF MEASUREMENTS.

Diameter of disc.

§ 22. Preliminary measurements of the disc while still mounted were made on August 11, 1882, with callipers by Messrs. ELLIOTT. Read by the vernier of the instrument itself the mean diameter was

$$2a = 310{\cdot}76 \text{ millims.}$$

The opening of the callipers was also determined independently by reference with the aid of microscopes to a verified scale of millimetres. In this way

$$2a = 310{\cdot}77 \text{ millims.}$$

The circumference was also measured by a steel tape, afterwards compared with the millimetre scale. Correction being made for the thickness of the tape, the result was

$$2a = 310{\cdot}84 \text{ millims.}$$

After the disc had been dismounted, the diameter could be determined more advantageously by direct observation through microscopes focussed upon its edge with subsequent reference to the standard scale. It was found (August 19, 1882) that a very appreciable difference existed between the diameter of the upper and lower faces, showing that the edge was somewhat conical. At the upper edge the diameter was 310·80, and at the lower edge 310·58. These were the extremes. At the middle of the thickness the diameter was 310·75. This departure from the truly cylindrical form was undoubtedly a defect in the apparatus, which could easily have been avoided if detected in time. When the apparatus was first set up, the success of the experiment was problematical, and a minute examination of the disc seemed premature. The diameter to be adopted is an average taken with reference to the conductivity of brush contact. The whole width of the brush being decidedly less than the thickness of the disc, and the pressure being greatest at the central parts, we decided (of course without knowing to what precise final result the estimate would lead) to take the mean of 310·75 and $\frac{1}{2}$ (310·58+310·80). Thus

$$2a = 310{\cdot}72 \text{ millims.}$$

The error due to the conicality of the edge cannot exceed one part in 5000 at the worst, and thus it appeared scarcely worth while to correct the defect and repeat the spinnings.

The diameter of the shaft at the place where the other brush contact was made, was found to be ·825 inch, or 20·96 millims.

The induction coils.

§ 23. These are the same as were used in Mr. GLAZEBROOK's measurement, and were wound by Professor CHRYSTAL in 1878. The following are the dimensions; for further particulars reference may be made to Mr. GLAZEBROOK's Memoir.*

	A	B.	Mean.
Mean radius in centims (A)	25·753	25·766	25·760
Radial width of section ($2h$)	1·92	1·90	1·91
Axial width of section ($2k$)	1·896	1·899	1·897
Number of windings.	797	791	$\frac{1}{2}\times 1588$
Resistance (approximate) in B A. units .	84	83	$\frac{1}{2}\times 167$

Since the coils are so nearly similar and were used symmetrically, it is sufficient to use the numbers in the last column. The section of the ring is shown in fig. 4 full size.

To find the distance of mean planes the following measurements of the thicknesses of the rims are required. They are given in centimetres.

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* Phil. Trans., 1883, p. 223.

	A.	B.
Rim (marked side)	·478	·446
Channel	1·896	1·899
Rim (unmarked side)	·488	·465
Total thickness of rim	2·862	2·810

Now that the rings are wound it is difficult to verify these numbers. However, the total thickness of the rings at the places touched by the distance pieces in the arrangement used for Series III. was taken, with the result

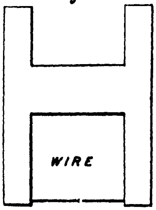
	A.	B.
Mean of three places.	2·8625	2·8067

These latter values of the thicknesses will be used in the calculation of Series III.

In Series I. and II. the rings were not reversed, and we must use the numbers above given for the thicknesses of rims which were contiguous to the slips of glass; but in this case the result is not at all sensitive to changes in the distance of mean planes. The rims contiguous to the glass were for both coils the *marked* rims, of which the aggregate thickness is ·924. If we add to this the thickness of the glass strips ·454, we obtain 1·378 as the distance between the wire sections. Again, adding the mean axial width of section 1·897, we find as the distance of the mean planes

$$2b = 3{\cdot}275 \text{ centims.}$$

Fig. 4



The distance-pieces.

§ 24. The measurement of the distance-pieces used for the third series was made with great care. As only the mean is required, the three pieces were held under the microscopes in one length by a nut and a long bolt running through Readings were taken in several positions, as the pieces were turned round, and reference was finally made to the standard scale. Two independent measurements gave 83·580 and 83·579, mean 83·5795 centims., as the aggregate length. This was further verified by measuring each piece separately with callipers, the sum of the lengths thus found being 83·582. For the mean length of these distance-pieces we take

$$27{\cdot}8598 \text{ centims.}$$

As has been already explained, the rings were used in two positions relatively to the distance-pieces, with the view of eliminating any uncertainty as to the situation of the mean planes, and of rendering the final result independent of all measurements of thickness except that of the total thicknesses of the rings. Thus the mean distance of mean planes in the two positions is

$$27{\cdot}8598+\tfrac{1}{2}(2{\cdot}8625+2{\cdot}8067)=30{\cdot}6944 \text{ centims.}$$

To compare the partial results for the two positions separately, we must use the thicknesses of the rims which were in contact with the distance pieces. In the first position these were the marked rims, and thus the distance of mean planes

$$=27{\cdot}860+{\cdot}478+{\cdot}446+1{\cdot}897=30{\cdot}681 \text{ centims.}$$

In like manner for the second position we find

$$27{\cdot}860+{\cdot}488+{\cdot}465+1{\cdot}897=30{\cdot}710 \text{ centims.}$$

The induction coefficients.

§ 25. Series I. and II. The distance (b) of the mean planes of the coils from the middle plane of the disc is

$$b=1{\cdot}637 \text{ centim.}$$

The extreme distances, required to be known for the quadrature, are

$$b+k=2{\cdot}585 \text{ centims.,} \quad b-k={\cdot}689 \text{ centim.}$$

The extreme and mean radii are

$$\text{A}-h=24{\cdot}805 \text{ centims.,} \quad \text{A}=25{\cdot}760 \text{ centims.,} \quad \text{A}+h=26{\cdot}715 \text{ centims.}$$

while

$$a=15{\cdot}536 \text{ centims.}$$

The coefficient of induction between the disc and the middle turn of the coil, denoted by $\text{M}(\text{A}, a, b)$, is equal to $4\pi\sqrt{(\text{A}a)}.f(\gamma)$, where $f(\gamma)$ is a function of γ given by tables.* The angle γ itself is defined by

$$\sin\gamma=\frac{2\sqrt{(\text{A}a)}}{\sqrt{\{(\text{A}+a)^2+b^2\}}}$$

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* MAXWELL'S 'Electricity and Magnetism,' 2nd edition, § 706.

It is not necessary to give the details of the calculations, which have been carefully checked. The tabular interval being 6′, it was found desirable in many cases to proceed beyond the simple interpolation by first differences. The results are

$$\begin{aligned} M(A, a, b)&=215{\cdot}4674\\ M(A+h, a, b)&=205{\cdot}1917\\ M(A-h, a, b)&=226{\cdot}9835\\ M(A, a, b+k)&=211{\cdot}7246\\ M(A, a, b-k)&=217{\cdot}5972 \end{aligned}$$

The mean coefficient for the area of the section is found by doubling the first of these values, adding in the others, and then dividing by 6.

Thus

$$M=215{\cdot}405$$[*]

The separate values allow us to form an estimate of the effect of errors in the fundamental data. If we write

$$\frac{dM}{M}=\lambda\frac{dA}{A}+\mu\frac{db}{b}+\nu\frac{da}{a},$$

we may take approximately

$$\lambda=\frac{M(A+h, a, b)-M(A-h, a, b)}{2h}\div\frac{M}{A}=-1{\cdot}36$$

In like manner, $\mu=-{\cdot}02$, whence, since $\lambda+\mu+\nu=1$, $\nu=+2{\cdot}38$.

Series III. In this case the data remain precisely as before, except that we now have $b=15{\cdot}3472$.[*]

We find

$$\begin{aligned} M(A, a, b)&=110{\cdot}9240\\ M(A+h, a, b)&=111{\cdot}2573\\ M(A-h, a, b)&=110{\cdot}2442\\ M(A, a, b+k)&=104{\cdot}5571\\ M(A, a, b-k)&=117{\cdot}6519 \end{aligned}$$

whence

$$M=110{\cdot}926$$

Determining λ, μ, ν, as in the former case, we find

$$\frac{dM}{M}=+{\cdot}123\frac{dA}{A}-{\cdot}956\frac{db}{b}+1{\cdot}833\frac{da}{a}.$$

From these values, calculated for the circumference of the disc, we have to subtract the value (M_0) applicable to the small circuit touched by the inner brush. The area of this is $\frac{1}{4}\pi(2{\cdot}096)^2$. For the first and second series we have

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[*] The factor expressing the number of windings is omitted.

$$M_0=\frac{2\pi A^2}{(A^2+b^2)^{\frac{3}{2}}}\frac{1}{4}\pi(2{\cdot}096)^2={\cdot}836$$

For the third series in like manner

$$M_0={\cdot}534$$

Thus finally for the first and second series

$$M-M_0=214{\cdot}569,$$

and for the third series

$$M-M_0=110{\cdot}392$$

The resistance coils.

§ 26. In all three series the resistance b, fig. 3, was a German-silver coil of about $\frac{1}{10}$, referred to for brevity as the $[\frac{1}{10}]$; and the resistance a was composed of three resistances in multiple arc, the first two being standard singles, and the third a resistance such as 7 B.A. units taken from a box. To make the necessary change, according to the plan already explained in § 9, the 7 would be replaced by 8. The value of a is of course determined principally by the unit resistance coils, and only secondarily by the resistance taken from the box.

The third element of the system of resistances was varied in the different series. In the first series c was a [10], in the second series it was [10]+[5]+[1], and in the third series [10]+[5]+[5′]. Besides the standard singles, whose values at various temperatures was already known in terms of the mean B.A. unit, we had to determine accurately the values of the $[\frac{1}{10}]$, the [10], the [5], and the [5′], as well as the small resistances of the various connecting pieces employed.

The [10] has been determined in various ways, but principally by means of the device referred to in the former paper.* Three German-silver wires of about 3 units each are wound on the same tube, and their terminals are so arranged that by means of a base board containing mercury cups they can be combined either in multiple arc or in series. In the former combination they are compared with a standard single, and the resistance is found to be (say) $1+\alpha$, where α is small. The coils are now without loss of time combined in series, a change which can be effected in a moment. The resistance in series is very approximately $9+9\alpha$; by the addition of the standard single it becomes $10+9\alpha$, and can now be compared with the [10]. If the difference observed be β we have $[10]=10+9\alpha+\beta$. By this method it is easy to obtain an accuracy of at least $\frac{1}{10000}$.

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* Phil. Trans., Part II., 1882, p. 697.

The [5]'s were determined in two ways. Five singles were combined in series and compared with one of the [5]'s; afterwards the two [5]'s were compared with one another. In the second method, which is probably preferable, the sum of the two [5]'s was found by comparison with the [10]. From the sum and difference the separate values can of course be deduced.

The measurement of the $[\frac{1}{10}]$ demanded some precaution on account of its smallness. Two standard singles, the [10], and the $\frac{1}{10}$, were combined with four insulated mercury cups, and without the use of connecting pieces, so as to form a WHEATSTONE's balance (fig. 5), care being taken to bring the associated battery and galvanometer terminals into immediate contact with the legs of the $[\frac{1}{10}]$ (see § 12). To get the means of adjustment, a box, giving resistances up to 10,000, was placed in multiple arc with one of the singles. If, as was the case, the four coils be so nearly in proportion that a resistance of several hundreds from the box is needed for balance, the delicacy of the arrangement is all that can be desired. Readings are taken also with battery reversed, to eliminate thermo-electric disturbances. Especial pains were taken with the measurement of the $[\frac{1}{10}]$, and of the [10], errors of which would be propagated into the results of all three series.

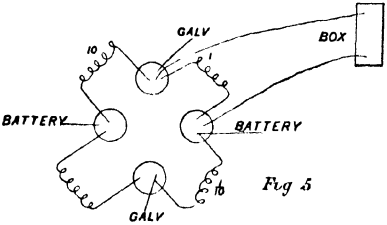

Fig 5

§ 27. The various temperatures of the coils at the time of use, and the fluctuations from day to day, complicate the calculation of the effective resistances R_1 and R_2, which in principle is simple enough. The results are given in column II. of the Tables. Thus in the first series on July 14, when the effective resistance was ·0044076 B.A., as calculated from the values of a, b, c, for the observed temperatures of the coils, the effect (E–W) of reversing the battery key (corrected for direct effect) was -30 divisions of the galvanometer scale, the direction of rotation being positive. When the effective resistance was altered to ·0044430, the difference E–W became $+10$ divisions. From these results we infer that E–W would vanish for the effective resistance ·0044341, as given in column V. The corresponding result with negative rotation is given in column VI. These resistances relate to the actual speed of rotation determined by the frequency of vibration of the electric fork (§ 6). To render the results of different days fairly comparable, two small corrections have to be introduced, the first relating to small alterations in the relative frequencies of the two forks, as shown by the number of beats per minute (column VII.), the second to variations in the frequency of the standard fork itself, dependent upon change of temperature. The temperatures were read by a thermometer which stood between the prongs of the standard, and are given in column IX. The corrections necessary for reduction to a standard number of beats (16 per minute) and to a standard temperature (16°) are tabulated in columns VIII. and X., and the corrected results themselves in XI. and XII. In all cases the electric fork vibrated *more* quickly than the standard.

The degree of accordance in the numbers entered in these columns shows the success of the observations, so far as relates to errors of a casual character. In column XIII. the results of the positive and negative rotations are combined, so as to exhibit the total result of the day's work.

The Table, showing the results of the third series, is divided into two parts, corresponding to the two positions of the induction coils, before and after reversal (§ 19). In each position, it will be seen that two sets of observations were taken upon one of the days. Both sets, however, were complete, and in the interval between them the resistance coils were all dismounted. A similar precaution was taken at least once in each of Series I. and II.

FIRST SERIES.

Coils near together.

Speed of disc about 12·8 revolutions per second.

Approximate resistances $a=\frac{1}{2}$, $b=\frac{1}{10}$, $c=10$.

Date.	Effective resistance (in B.A. units) used.	Difference of reading of galvanometer on reversal of current.		Effective resistance (in B A units) corresponding to zero difference in galvanometer.		Correction for change of speed of fork.				Effective resistance (in B.A. units) as finally corrected.		Means of resistances with both directions of rotation.
July, 1882.		Rotation +	Rotation −	Rotation +	Rotation −	Beats between forks.	Correction to 16 beats.	Temperature of standard fork.	Correction to 16°.	Rotation +	Rotation −	
14th	·0044076	−30·0	+32·0	0044341	·0044371	17	−·0000006	° 16·7	+·0000003	·0044338	·0044368	00443530
„	·0044430	+10·0	− 6·3									
15th	·0044084	−24·6	+25·8	0044333	0044346	16	0	17·1	+·0000005	·0044338	·0044351	00443445
„	·0044438	+10·5	− 9·2									
17th	·0044090	−25·4	+27·0	·0044322	·0044315	15	+·0000006	17·3	+ 0000006	·0044334	·0044327	00443305
„	·0044444	+13·4	−15·5									
18th	·0044095	−27·4	+25·9	0044331	·0044332	16	0	17·6	+·0000008	0044339	·0044340	00443395
„	·0044449	+13·7	−12·8									
19th	·0044100	−23·9	+27·4	·0044324	0044337	16	0	18·0	+·0000010	0044334	·0044347	·00443405
„	·0044454	+13·9	−13·5									
20th	·0044100	−24·4	+25·4	0044326	0044328	16	0	17·9	+ 0000009	·0044335	0044337	·00443360
„	·0044454	+13·8	−14·0									
									Means . .	·00443363	·00443450	·00443407

SECOND Series.

Coils near together.
Speed of disc about 8 revolutions per second.
Approximate resistances $a=\frac{1}{2}$, $b=\frac{1}{10}$, $c=16$.

Date.	Effective resistance (in B.A. units) used.	Difference of reading of galvanometer on reversal of current.		Effective resistance (in B A units) corresponding to zero difference in galvanometer.		Correction for change of speed of fork.				Effective resistance (in B.A. units) as finally corrected.		Means of effective resistances with both directions of rotation.
August, 1882.		Rotation +	Rotation −	Rotation +	Rotation −	Beats between forks.	Correction to 72 beats.	Temperature of standard fork.	Correction to 16°.	Rotation +	Rotation −	
7th "	·0027827 ·0028126	− 8·2 +20·1	+ 9·4 −21·5	·0027914	·0027918	73	−·0000004	17·6°	+·0000005	·0027915	·0027919	·00279170
8th "	·0027821 ·0028120	− 8·6 +21·0	+ 9·5 −19·1	·0027908	·0027920	73	−·0000004	17·2	+·0000004	·0027908	·0027920	·00279140
9th "	·0027826 ·0028125	− 7·9 +19·5	+ 7·5 −19·1	·0027912	·0027910	72	0	17·5	+·0000005	·0027917	·0027915	·00279160
									Means . .	·00279133	·00279180	·00279157

THIRD Series.

Coils separated.

Speed of disc about 12·8 revolutions per second.

Approximate resistances $a=\frac{1}{2}$, $b=\frac{1}{10}$, $c=20$.

Date.	Effective resistance in B A. units used.	Difference of reading of galvanometer on reversal of current.		Effective resistance (in B A. units) corresponding to zero difference in galvanometer.		Correction for change of speed of fork.				Effective resistance (in B.A. units) as finally corrected.		Means for each day of observations with both directions of rotation.	Means of all the observations with each direction of rotation.		Means of all observations with both directions of rotation.
August, 1882.		Rotation +	Rotation −	Rotation +	Rotation −	Beats.	Correction to 72 beats.	Temperature of standard fork	Correction to 16°.	Rotation +	Rotation −		Rotation +	Rotation −	
14th	·0022981	+ ·1	− ·3	·0022982	·0022985	73	−·0000003	18 4	+·0000006	0022985	0022988	·00229865	·00229853	·00229910	00229881
„	·0023100	−11·0	+ 9·1												
15th	·0022976	− ·1	− 1·9	·0022975	·0022995	72	0	18·4	+ 0000006	·0022981	·0023001	00229910			
„	·0023095	−11·7	+10·0												
15th	·0022973	+ 1 1	− ·6	0022985	·0022979	72	0	18 1	+·0000005	·0022990	·0022984	00229870			
„	·0023092	− 9 9	+11 1												
Second position of induction coils.															
16th	·0022969	− ·3	0	·0022966	·0022969	72	0	17·6	+ 0000004	·0022970	·0022973	00229715	·00229633	·00229650	·00229642
„	·0023088	−11·9	+ 7·8												
17th	·0022965	− ·8	+ ·7	·0022956	·0022958	72	0	17·4	+·0000004	·0022960	·0022962	·00229610			
„	·0023084	−11·2	+12·1												
17th	·0022966	− ·8	+ 1·0	·0022956	·0022956	72	0	17·4	+·0000004	·0022960	·0022960	00229600			
„	·0023085	−10 2	+12·4												
												Means.	·00229743	·00229780	·00229762

§ 28. The results given in these tables are the effective resistances required to obtain a balance, expressed in terms of the B.A. unit. To reduce them to absolute measure we must multiply by 10^9, and by a factor, which we may call x, expressing the absolute value of the B.A. unit in terms of 10^9, and which it is our object to determine.

The actual value of the same quantities in absolute measure is found by multiplying the coefficients of induction $(M-M_0)$ already given (§ 25), by the number of turns in the coils 1588, and by the number of revolutions per second.

In the first series the frequency of vibration (f) of the electric tuning-fork was in the standard case (see Appendix)

$$f=\tfrac{1}{2}(128{\cdot}140+\tfrac{16}{60})=\tfrac{1}{2}\times 128{\cdot}407$$

and the number of revolutions per second is equal to $2f\div 10$. In the second and third series $2f=129{\cdot}340$, a number which in the second series is to be divided by 16, and in the third series by 10, in order to obtain the number of revolutions per second.

The equation to determine x is thus for the first series of observations

$$214{\cdot}569\times 1588\times 12{\cdot}8407=x\times {\cdot}00443407\times 10^9,$$

whence

$$x={\cdot}98674.$$

From the second series

$$x=\frac{214{\cdot}569\times 1588\times 129\ 340}{16\times 10^9\times {\cdot}00279157}={\cdot}98669$$

From the third series

$$x=\frac{110{\cdot}392\times 1588\times 129{\cdot}340}{10\times 10^9\times {\cdot}00229762}={\cdot}98683$$

These are the final results already considered in § 21.

APPENDIX.

Frequency of Vibration of Standard Fork.

All our measurements, both by this method and by that of the revolving coil, being dependent upon the pitch of a standard tuning-fork, we have considered it advisable to determine this element afresh. As in the first determination,[*] a fork vibrating about 32 times per second rendered intermittent an electric current, which, passing through the coils of small electromagnets, maintained in vibration not only the interrupter fork itself, but also a second fork of pitch about 128. After the apparatus has been a short time in operation, the vibrations of the second fork are exactly four times as quick as those of the first, independently of any precise tuning; and they give rise to audible beats when the standard fork is simultaneously excited. In the presence of extraneous noises the observation of the beats is much facilitated by the use of resonators, with one of which the ear may be connected by an indiarubber tube. The object to be aimed at is to make the intensities of the two sounds (as they reach the ear) very nearly equal. The moment of antagonism is then marked by a well defined silence, whose occurrence can be timed to within a second, although the whole duration of the beat may be 20 seconds or more. Without fresh bowing of the standard, the silences can be observed satisfactorily for at least a minute.

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[*] Proc. Roy. Soc, May, 1881, p. 137.

In the first determination the comparison between the fork of frequency 32, and the pendulum of the clock was made directly. The observer, looking over a plate carried by the upper prong of the fork, obtained 32 views per second, *i.e.*, 64 views of the pendulum in one complete vibration. The immediate subject of observation is a silvered bead attached to the bottom of the pendulum, upon which as it passes the position of equilibrium the light of a paraffin lamp is concentrated. Close in front of the pendulum is placed a screen perforated by a somewhat narrow vertical slit. If the period of the pendulum were a precise multiple of that of the fork, the flash of light which to ordinary observation would be visible at each passage, would either be visible, or be obscured, in a permanent manner. If, as in practice, the coincidence be not perfect, the flashes appear and disappear in a regular cycle, whose period is the time in which the fork gains (or loses) one complete vibration This period can be determined with any degree of precision by a sufficient prolongation of the observations.

On account of the large number of views per second, the interval between successive visible positions of the bead, even when it is moving with maximum velocity, is rather small; and thus the adjustment of the apparatus is somewhat delicate.* In order to meet this objection, a modification has been introduced, which must now be explained.†

A few years ago it was shown almost simultaneously by LA COUR and by Lord RAYLEIGH, that an electromagnetic engine could be accurately governed by an interrupter-fork. The construction (fig. 6) which has been found most suitable is similar to that of FROMENT'S engine. A horizontal shaft revolving upon steel points carries a number of parallel soft iron armatures, disposed symmetrically round the circumference. In the course of the revolution these armatures pass in succession between the poles of a vertical horse-shoe electromagnet, so as almost to complete the magnetic circuit. It is much better that the armatures should pass *between* the poles than *over* them, as in the most usual arrangement, for in the latter case the bearings are subjected to an unnecessary and prejudicial strain. The wheel may be used either with or without an independent driving power. In the former case the power should be very steady, and adjusted so as to give by itself nearly the speed intended. The currents from the interrupter-fork are passed also through the electromagnet of the engine, and give the force required to accelerate or retard the motion so that it may exactly synchronise with the fork, one armature passing for each complete vibration. If the independent power is in excess, the phase of the motion is such that the electromagnet is excited principally after the armatures have passed through the electromagnet; if the independent power is in defect, the electromagnet is excited principally while the armatures are approaching it. Within certain limits any necessary acceleration or retardation is obtained by suitable self-acting adjustment of phase.

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* In the earliest use of this method ("Nature," xvii., p. 12, 1877) the break-fork had a frequency of about 13, and no difficulty of this kind was experienced.

† [July, 1883.—It should be stated, however, that the wheel may easily be dispensed with, if proper care be taken in the illumination of the bead and in the management of the fork. The vibration should be vigorous, and the screens so arranged that the view past the fork at the moment of greatest elongation should be of short duration. Determinations by this method (without the wheel) have often been made successfully by students in the Cavendish Laboratory.]

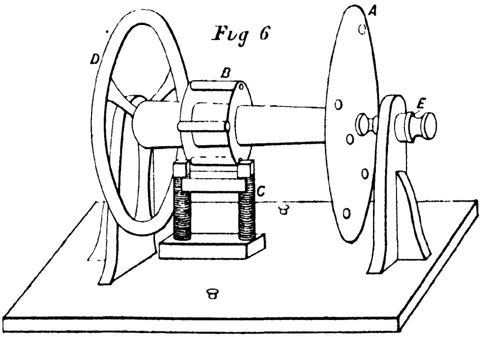

Fig 6

If when the wheel is moving steadily under the influence of the intermittent currents, a slight disturbance is communicated to it, oscillations will set in, the wheel being alternately in advance and in the rear of its proper position. In some cases these oscillations are very persistent, and interfere seriously with the utility of the instrument. To check them, a hollow ring filled with water is attached to the shaft, and revolves with it. When the rotation is perfectly regular, the water behaves as if it were a rigid body and offers no impediment to the motion, but it tends to check variations of speed of moderate period. The oscillations, when they exist, are usually audible; and in any case the behaviour of the wheel in this and other respects may be examined by looking at the interrupter-fork through a paper disc carried by the wheel and perforated symmetrically along a circle with holes equally numerous with the armatures. When all is regular, the prongs of the fork are seen in one phase only, so long as the eye retains a position fixed in space.

When the wheel runs lightly, independent driving power may be dispensed with, a sufficient amount of work being obtainable from the intermittent governing current. In the present case the whole apparatus, consisting of the two forks and the wheel, was driven by one current supplied from three GROVE cells. The only difficulty experienced is in starting the wheel. By means of string passed once round the shaft, alternately tightened for the advance and slackened for the return, it is easy to cause the wheel to achieve a speed in excess of the necessary eight revolutions per second. But it will not usually happen, every time the speed falls through the proper value, that the wheel will engage with the fork. For this purpose it is necessary that at the moment in question the phase of the wheel should be correct, within limits, which may be narrow when there is no great margin of power; and this can only happen by chance. Several attempts may be necessary before success is reached. With a little practice, however, there is no great loss of time, the ear learning to recognise, by the gradual slowing and subsequent quickening of a sort of beat, when the wheel has passed through the right speed without engagement. A fresh impulse is then given without waiting further. After a start is once effected, the wheel will usually run, keeping perfect time with the fork, until the battery is exhausted.

The wheel employed in the experiments we are now concerned with, has *four* soft iron armatures, and is governed by the interrupter-fork of frequency 32. The speed of the wheel is thus eight revolutions per second; and a single hole in a paper disc carried round with it allows eight views of the pendulum per second, the smallest number of views obtainable by direct use of the fork being 32. Altogether we may regard the frequency of the interrupter-fork as being multiplied four times precisely in the frequency of the auxiliary fork, and as divided four times precisely in the frequency of the wheel. The former is directly comparable with the standard fork, and the latter with the clock. The standard fork was screwed to the table precisely as during the electrical measurements. A thermometer placed between the prongs gave the temperature with fair accuracy.

The calculation of the results is very simple. Supposing in the first instance that the clock is correct, let a be the number of cycles per second (perhaps $\frac{1}{40}$) between the wheel and the clock. Since the period of a cycle is the time required for the wheel to gain, or to lose, one revolution upon the clock, the frequency of revolution is $8 \pm a$. The frequency of the auxiliary fork is precisely 16 times as great, *i.e.*, $128 \pm 16a$. If b be the number of beats per second between the two forks, the frequency of the standard is

$$128 \pm 16a \pm b$$

To give an idea of the magnitudes of the numbers concerned, it will be advisable to quote in detail the results of one day's observations. On October 19, with a certain loading of the interrupter-fork, the cycle of the pendulum occupied about 78 seconds, and the beats were at the rate of about six per minute. The interrupter was then *sharpened*, after which several observations were taken of the duration of five cycles of the pendulum, and of 16 beats between the forks. For the former the times found were 210, 210, 212 seconds; for the latter by simultaneous observation 58, 58, 59, 59, 59, 60, 60 seconds. The temperature, as given by the thermometer, ranged from 17°·2 to 17°·4. After the sharpening of the interrupter, the frequency both of the wheel and of the auxiliary fork was increased, so that the sign of $16a$ in the expression written above is determined to be + and that of b to be —. Using the mean values we find

$$16a = \cdot 3797, \qquad b = \cdot 2712$$

whence

$$128 + 16a - b = 128 \cdot 108$$

To this we must add ·009, making altogether 128·117, to allow for the gaining rate of the clock, which was 6½ seconds per diem. This corresponds to a mean temperature 17°·3.

The procedure adopted was quite good enough for our purpose; but if it were desired to push the power of the method to its limit, the work should be undertaken at an astronomical observatory, and extended over the whole time required to rate the clock by observations of the stars. In this way the comparison of the period of vibration of the standard fork with the mean solar second could be effected with the same degree of accuracy as that to which the former quantity is capable of definition. Without this precaution we cannot be quite sure that the rate of the clock at the time of the observations is identical with the mean rate employed in the calculation. It is scarcely necessary to say that the uncertainty which arises under this head is common to every method by which absolute pitch could be determined.

The results obtained, including those recorded previously,* are given in the accompanying table. They are well represented by the formula

$$128 \cdot 140 \times \{1 - (t - 16)^0 \times \cdot 00011\},$$

in which the temperature coefficient used (·00011) is that found by M'Leod and Clarke.† The numbers in the fourth column are calculated from the formula.

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* Proc. Roy. Soc., May, 1881, p. 138.

† Phil. Trans., Part I., 1880.

Date.	Temperature.	Frequency by observation.	Frequency by calculation
1881	13°	128·180	128 182
1881	14°·6	128 161	128 160
October, 1882	15° 98	128 141	128 140
October, 1882	17° 45	128 122	128 120
October, 1882	17°·6	128 119	128 118
October, 1882	17° 3	128 117	128 122

Of the small discrepancies which the table exhibits it is probable that the larger part is due to imperfect knowledge of the actual temperatures of the standard fork. The use of screens to cut off radiation from the observers would probably have effected an improvement. For the highest accuracy some sort of jacket, or chamber, would have to be contrived.

SECOND APPENDIX.

(Added July, 1883.)

On the Effect of the Imperfect Insulation of Coils.

In a former paper (Phil. Trans., 1882, Part II.) it was pointed out that the method of the revolving coil, employed by the first B.A. Committee, possesses the important advantage that it is possible to detect the existence of leakage from turn to turn, or from layer to layer, of the coil of wire. The general influence of such leakage, if undetected, upon the final number x expressing the ratio of the resistance of the coil when measured (R) in absolute units to its resistance $r\times 10^9$ as referred to B.A. units, is easily seen by supposing that one turn of the coil is simply short-circuited. The formula in C.G.S. measure is

$$x=\frac{R}{r\times 10^9}=\frac{\pi^2 n^2 a\,\omega\cot\phi}{r\times 10^9} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

During the revolutions the short circuited turn produces its full effect in deflecting the magnet, and error arises only in the comparison with the standard of resistance. The quantity r will evidently be under-estimated by $1/n$, and this will lead to an *over*-estimate of x, also by $1/n$. This result, however, is modified, if as in practice we take only the *difference* of effects observed when the wire contact is open and closed. The short-circuited turn will produce its effect in *both* cases, and its influence will therefore disappear from the result. For all purposes it will be virtually non-existent, and the error produced is the same as if n had simply been miscounted. The final number x will thus be over-estimated by the fraction $2/n$.

In LORENTZ's method the effect of a short circuit in the induction coil is in the same direction. M, and therefore R and x, will be over-estimated by $1/n$.

If we examine the formulæ applicable to determinations by other methods, we shall see that a similar conclusion holds good, so that in every case leakage leads to an over-valuation of x, at least whenever the result is calculated from the number of turns of wire in a coil.* Even without such an examination, it is pretty evident from consideration of the magnitudes involved that the large factor 10^9 in the denominator of the formula corresponding to (1) can only be compensated by one or more large factors expressive of the number of windings in a coil or coils. An over-valuation of these factors, due to leakage, will therefore lead to an over-valuation of x.

In carefully constructed coils serious leakage is, perhaps, not likely to occur, but its presence in a smaller degree is more probable, and is usually difficult of detection. So far as this argument applies, we may say that the smaller values of the number expressive of the B.A. unit, or of the mercury unit, in absolute measure are to be preferred to the larger.

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* The case is different when the constants of a coil of many turns are determined by electrical comparison, as for instance in KOHLRAUSCH's recent correction of the constant of his earth-inductor.

VIII. *On* ABEL'S *Theorem and Abelian Functions.*

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THE present paper is divided into two sections. The object of Section I. is to obtain an expression for an integral more general than, but intimately connected with, that occurring in ABEL'S theorem. The latter, as enunciated by Professor ROWE in his memoir in the Phil. Trans., 1881, is as follows:—If

$$\chi(x, y)=0$$

be a rational algebraical equation between x and y, then an expression can always be found for

$$\Sigma\int\frac{\mathrm{U}dx}{f(x)\frac{\delta\chi}{\delta y}}$$

where $f(x)$ is a function of x only, U a rational algebraical integral function of x and y, and the upper limits of the series of integrals are the roots of the eliminant with regard to y of $\chi(x, y)=0$ and a function $\theta(x, y)$.

In the case here considered two equations respectively of the degrees m and n between three variables

$$\mathrm{F}_m(x, y, z)=0$$
$$\mathrm{F}_n(x, y, z)=0$$

are given (these alone being considered, as subsequent generalisation to the case of r equations between r dependent variables and one independent variable is obvious); and an expression is obtained for

$$\Sigma\int\frac{\mathrm{U}dx}{f(x)\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}$$

the upper limits of the integrals being given by the roots of the equation arrived at by eliminating y and z between F_m, F_n and an arbitrary equation

$$\mathrm{F}_p(x, y, z)=0$$

or, what is the same thing, by the co-ordinates x of the points of intersection of the three surfaces represented by F_m, F_n, F_p.

Some preliminary considerations (in connexion with §§ 92 *sqq.* of SALMON's 'Higher Algebra') are adduced in reference to the eliminants of the three equations in each of the variables; thus if X be the equation in x obtained by eliminating y and z, it is expressed in the form

$$X = B_m F_m + B_n F_n + B_p F_p$$

which afterwards proves useful. Then the ordinary case (above referred to) of ABEL's theorem is treated on the lines laid down in CLEBSCH and GORDAN's 'Treatise on the Abelian Functions;' and under the guidance of this the more general form is investigated with the result

$$\Sigma \int \frac{U dx}{f(x) J \left(\dfrac{F_m, F_n}{y, z} \right)} = \Theta \left[\frac{1}{f(x)} \right] \Sigma \left\{ \frac{U}{J} \log F_p \right\} + C$$

Θ being the symbol introduced by BOOLE.

The remainder of this section is occupied with the discussion of two examples of this theorem. In the first, expressions are obtained for $E(u_1 + u_2 + u_3)$ and $\Pi(u_1 + u_2 + u_3)$, E and Π being the second and third elliptic integrals; and in the second example $E(u_1 + u_2 + \ldots + u_7)$ is considered.

In Section II. the addition-theorem for the functions presented in WEIERSTRASS's memoir ('Crelle,' t. lii., (1856), p. 285) is investigated. [It may be pointed out that the fundamental equations occur as natural examples of the more general form of ABEL's theorem proved in Section I.; but the equations which are obtained almost immediately are identical with those used by WEIERSTRASS, and so this case does not belong distinctively to the form of ABEL's theorem for the curve of double curvature.] For the purpose of the section use is made of the "integral-function," the partial differential coefficients of which with respect to the amplitudes give the squares of the Abelian functions. The theory is worked out at some length, and the necessary formulæ are deduced from the fundamental equations in a manner somewhat different from that of WEIERSTRASS. From the form first obtained for the sum of three integral-functions an important theorem is deduced in § 21, and a verification of this is afterwards furnished by the expansion of the two sides of the equation. It is then applied, as already mentioned, to obtain the addition-theorem for the functions.

In §§ 25, 26 is given the discussion of a particular case of the above, viz., when the functions are of the order 2, the fifteen functions being the quotients of all but one of the double theta-functions by that one. This has already formed the subject of a paper by CAYLEY in 'Crelle,' t. lxxxviii. (1878), p. 74.

SECTION I.

1. Before proceeding to the consideration of the theorem it is necessary to indicate the form in which the eliminant of three equations in three variables (or in general of μ equations in μ variables) will be used.

If we consider two equations in two variables say

$$f_n \equiv x_0 + x_1 y + x_2 y^2 + \dots + x_n y^n = 0$$
$$\phi_m \equiv X_0 + X_1 y + X_2 y^2 + \dots + X_m y^m = 0$$

and if X be the eliminant of f_n and ϕ_m with regard to y, then we have

$$X = Af + B\phi.$$

Now X being of degree mn in x, A must be of degree $mn-n$ and B of $mn-m$; while it is sufficient that the highest power of y in A be the $(m-1)^{\text{th}}$ and in B the $(n-1)^{\text{th}}$. Write then

$$A = A_0 + A_1 y + A_2 y^2 + \dots + A_{m-1} y^{m-1}$$
$$B = B_0 + B_1 y + B_2 y^2 + \dots + B_{n-1} y^{n-1}.$$

Substitute in X; since X is explicitly free from y all the coefficients of powers of y in the result must be zero. This then gives

$$\begin{array}{lll}
A_0x_1 + A_1x_0 & + B_0X_1 + B_1X_0 & = 0 \\
A_0x_2 + A_1x_1 + A_2x_0 & + B_0X_2 + B_1X_1 + B_2X_0 & = 0 \\
A_0x_3 + A_1x_2 + A_2x_1 + A_3x_0 & + B_0X_3 + B_1X_2 + B_2X_1 + B_3X_0 & = 0 \\
\dots\dots & \dots &
\end{array}$$

$m+n-1$ equations to determine the ratios of the $m+n$ quantities A, B. Let

$$E = \begin{vmatrix}
1 & y & y^2 & \dots & y^{m-1} & 0 & 0 & \dots & 0 \\
x_1 & x_0 & 0 & & 0 & X_1 & X_0 & \dots & \\
x_2 & x_1 & x_0 & \dots & 0 & X_2 & X_1 & X_0 \dots & \\
 & \cdot & & & & \cdot & \cdot & &
\end{vmatrix}$$

$$F = \begin{vmatrix}
0 & 0 \dots & & & 0 & 1 & y & y^2 \dots & y^{n-1} \\
x_1 & x_0 & 0 \dots & & & X_1 & X_0 & 0 \dots & \\
x_2 & x_1 & x_0 \dots & & & X_2 & X_1 & X_0 \,. & \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdot \cdot &
\end{vmatrix}$$

Then A_r bears to the minor of y^r in E the same ratio as B_s bears to the minor of y^s in F: thus

$$\frac{A}{E} = \frac{B}{F}.$$

But the diagonal term in E is

$$x_0^{m-1} X_m^n$$

and E is therefore of the degree $n(m-1)$, *i.e.*, of the same degree in X as A; it is obviously of the same degree in y; hence $\frac{A}{E}$ is merely an arithmetical constant, and we may write

$$A=E, \quad B=F.$$

2. When we come to apply this method to the formation of the eliminant with regard to y and z of three equations F_1, F_2, F_3 in three variables x, y, z the result, though of similar form, viz.:

$$A_{x,1}F_1+A_{x,2}F_2+A_{x,3}F_3,$$

can in general be obtained neither so directly nor without the help of the considerations in SALMON'S 'Higher Algebra,' §§ 92, *sqq.* If the three equations be each of the degree 2, the method will apply exactly as in the preceding paragraph and we obtain

$$X=A_{r,1}F_1+A_{x,2}F_2+A_{x,3}F_3$$

where

$$A_{x,r}=\alpha_{x,r}+\beta_{x,r}x+\gamma_{x,r}y;$$

but if the equations be not of this degree, then the following is our rule. Let F_1, F_2, F_3 be of the degrees m, n, p respectively: then we form all possible equations, which the variables satisfy, of degree not higher than $m+n+p-2$: thus we multiply F_1 by

$$y^{n+p-2},\ y^{n+p-3},\ \ldots,\ y^{n+p-3}z,\ y^{n+p-4}z,\ \ldots,\ y^{n+p-4}z^2,\ y^{n+p-5}z^2,\ \ldots$$

and so on; and so we obtain

$$\tfrac{1}{2}(n+p-1)(n+p)+\tfrac{1}{2}(p+m-1)(p+m)+\tfrac{1}{2}(m+n)(m+n-1)$$

equations from which to eliminate

$$\tfrac{1}{2}(m+n+p-1)(m+n+p)$$

quantities. But these equations are not all independent, being connected by a number of identities of the form

$$z^r y^s F_1.F_2=z^r y^s F_2.F_1$$

(where $r+s\leqq p-2$), of which there are $\frac{1}{2}p(p-1)$; there are $\frac{1}{2}n(n-1)$ of the form

$$z^{r'}y^{s'}F_1.F_3=z^{r'}y^{s'}F_3.F_1 \text{ where } r'+s'\leqq n-2$$

and $\frac{1}{2}m(m-1)$

$$z^{r''}y^{s''}F_2.F_3=z^{r''}y^{s''}F_3.F_2 \text{ where } r''+s''\leqq m-2,$$

and thus we have the proper number of equations. To find the eliminant X we write down the coefficients (which are, of course, functions of x) in the $\frac{1}{2}(n+p-1)(n+p)+\ldots$ equations; and reduce them to the form of a determinant by adding the coefficients in the $\frac{1}{2}p(p-1)+\ldots$ equations; then the eliminant is the quotient of the determinant formed by any

$$\tfrac{1}{2}(m+n+p-1)(m+n+p)$$

rows of the set first written down by the determinant formed from the second set after the elision of these rows.

3. To show how this can be brought into the desired form the easiest plan will be to consider an example. Let

$$F_1 \equiv Ay+Bz+C$$
$$F_2 \equiv A'y^2+F'yz+B'z^2+E'y+D'z+C'$$
$$F_3 \equiv A''y^3+D''y^2z+E''yz^2+B''z^3+F''y^2+G''yz+J''z^2+H''y+K''z+C''$$

where the coefficient of the highest powers of z and y are constants and those of other powers are functions of x such as make the order of the highest expression in the term of the same order as the equation; thus, for instance

$$F''=fx+f'$$
$$H''=hx^2+h'x+h''$$

and so on. Then we have, since $m=1$, $n=2$, $p=3$

	y^4	y^3z	yz^2	yz^3	z^4	y^3	yz	yz^2	z^3	y^2	yz	z^2	y	z	1	$yF_2\,F_1$	$zF_2.F_1$	$F_2\,F_1$	$F_3\,F_1$
y^3F_1	A	B				C										A'			A''
y^2zF_1		A	B				C									F'	A'		D''
yz^2F_1			A	B				C								B'	F'		E''
z^3F_1				A	B				C								B'		B''
y^2F_1						A	B			C						E'		A'	F''
yzF_1							A	B			C					D'	E'	F'	G''
z^2F_1								A	B			C					D'	B'	J''
yF_1										A	B		C			C'		E'	H''
zF_1											A	B		C			C'	D'	K''
F_1													A	B	C			C'	C''
y^2F_2	A'	F'	B'			E'	D'			C'						A			
yzF_2		A'	F'	B'			E'	D'			C'					B	A		
z^2F_2			A'	F'	B'			E'	D'			C'					B		
yF_2						A'	F'	B'		E'	D'		C'			C		A	
zF_2							A'	F'	B'		E'	D'		C'			C	B	
F_2										A'	F'	B'	E'	D'	C'			C	
yF_3	A''	D''	E''	B''		F''	G''	J''		H''	K''		C''						A
zF_3		A''	D''	E''	B''		F''	G''	J''		H''	K''		C''					B
F_3						A''	D''	E''	B''	F''	G''	J''	H''	K''	C''				C

To find the eliminant we choose any 15 rows (leaving out say the y^2F_1, yzF_1, zF_2, yF_3) and form a determinant, and then divide by

$$\begin{vmatrix} E' & 0 & A' & F'' \\ D' & E' & F' & G'' \\ 0 & C & B & 0 \\ 0 & 0 & 0 & A \end{vmatrix} = u.$$

(As the object is to illustrate the general case and not merely to get the result in this particular case we have not selected those rows which leave the denominator in the simplest form.) In the determinant of 15^2 constituents multiply each column by the quantity which stands at the head of it, add the results horizontally along all the rows, and replace the constituents of the last column by these new constituents which are, in order,

$$y^3F_1,\ y^2zF_1,\ yz^2F_1,\ z^3F_1,\ z^2F_1,\ yF_1,\ zF_1,\ F_1,\ y^2F_2,\ yzF_2,\ z^2F_2,\ yF_2,\ F_2,\ zF_3,\ F_3,$$

so that if we expand we have the numerator of our eliminant in the form

$$A'_{x,1}F_1 + A'_{x,2}F_2 + A'_{x,3}F_3$$

where the A's are determinants differing from the initial determinants in the last column alone; $A'_{x,1}$ has for its constituents there the coefficients of F_1 so long as F_1 occurs in the later form and then zeros; $A'_{x,2}$ those of F_2 where it occurs and elsewhere zeros; $A'_{x,3}$ those of F_3 where it occurs and elsewhere zeros. Moreover, we know that our eliminant is an integral function of x not extending in an infinite series; hence each of the coefficients A' must be divisible by u. If not, one of the F's (say F_1) must be so divisible; since u is a function of x only it follows that, when $u=0$, $F_1=0$ whatever z and y may be. We shall assume that such factors are removed before the investigation begins as they are useless for the purposes for which the functions are required; and hence we obtain our eliminant in the form

$$X = A_{x,1}F_1 + A_{x,2}F_2 + A_{x,3}F_3.$$

Similar remarks of course apply to Y and Z, the eliminants with regard to z and x, y and x.

4. We may also obtain the result as follows:

Between $F_m=0$, $F_p=0$ eliminate z and denote the eliminant by X_y; then, as we have already seen, X_y can be expressed in the form

$$X_y = \lambda_m F_m + \lambda_p F_p.$$

Between $F_n=0$, $F_p=0$ eliminate z and denote this eliminant by X'_y; then

$$X'_y = \mu_n F_n + \mu_p F_p.$$

Now X_y, X'_y are both functions of x and y; eliminating y between them and denoting the eliminant by X we have

$$X = \rho_y X_y + \rho'_y X'_y$$
$$= A_m F_m + A_n F_n + A_p F_p$$

of the same form as before.

This method of expressing an eliminant obviously admits of generalisation to the case of r equations in r variables.

5. The preceding method enables us to obtain the eliminants of the equations with regard to the different variables in a particular form which is useful in the proof of the general theorem in § 7; but when the object is merely to obtain the equation giving the roots x_μ which are to form the upper limits of our integrals we should arrive at the result more easily as follows.

Obviously

$$X = \prod_{\mu=1}^{\mu=mn} F_p(x, y_\mu, z_\mu)$$

where y_μ, z_μ constitute one of the mn pairs of roots of the equations

$$F_m = 0 \quad F_n = 0$$

regarded as giving y, z in terms of x; and the product is taken over all these pairs. Now the coefficients on the right-hand side will be symmetric functions of y and z, and these can be evaluated (by the method given in SALMON'S 'Higher Algebra,' § 74) in terms of x; and there will be obtained the required equation in x.

ABEL'S *Theorem.*

6. Let

$$\chi(x, y) = 0 \quad . \; . \quad\quad . \; . \quad\quad . \; . \; . \; . \qquad \text{(i)}$$

be an equation of the degree n which gives y in terms of x; and let

$$\theta(x, y)$$

denote a function of x of degree m—reducible to degree $n-1$ at most in y by means of (i)—the coefficients of y in which are functions of x and contain any number of arbitrary constants. Treating $\chi=0$, $\theta=0$ as two equations to determine the values of the variables, these arbitrary constants will enter into the expressions for the values of x, and will therefore vary when the latter vary. Let such a variation take place, so that

$$\frac{\delta\theta}{\delta x}dx + \frac{\delta\theta}{\delta y}dy + \delta\theta = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(ii)}$$

δ operating only on the constants in θ. Moreover we have from (i)

$$\frac{\partial\chi}{\partial x}dx+\frac{\partial\chi}{\partial y}dy=0$$

or writing

$$dw=\frac{dx}{\dfrac{\partial\chi}{\partial y}}=-\frac{dy}{\dfrac{\partial\chi}{\partial x}}$$

the equation (ii) becomes

$$-dw\left(\frac{\partial\chi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\theta}{\partial x}\frac{\partial\chi}{\partial y}\right)+\delta\theta=0$$

or

$$dw=\frac{\delta\theta}{\mathrm{J}(\chi,\theta)}$$

and therefore

$$\Sigma\mathrm{U}dw=\Sigma\frac{\mathrm{U}}{\mathrm{J}(\chi,\theta)}\delta\theta\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

where U is any rational function of x and y, and the summation is taken over all the roots x_μ of the equation obtained by the elimination of y between χ and θ.

Let X, Y respectively denote the eliminants of χ, θ with regard to y, x; then we can express X, Y in the form

$$\left.\begin{aligned}\mathrm{X}&=\mathrm{A}\chi+\mathrm{B}\theta\\ \mathrm{Y}&=\mathrm{C}\chi+\mathrm{D}\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

and we write

$$\Delta=\mathrm{AD}-\mathrm{BC}.$$

Now whatever the function U may be it can be written in the form

$$\frac{\mathrm{T}}{f(x)},$$

for it must be expressible as

$$\frac{f_2(x,y_1)}{f_1(x,y_1)}$$

that is,

$$\frac{f_2(x,y_1)\prod\limits_{\mu=2}^{\mu=n}f_1(x,y_\mu)}{\prod\limits_{\mu=1}^{\mu=n}f_1(x,y_\mu)}$$

which by means of the equation $\chi=0$ is at once reduced to the above form; thus

$$\Sigma\frac{\mathrm{T}}{f(x)}\frac{dx}{\dfrac{\partial\chi}{\partial y}}=\Sigma\frac{\mathrm{T}}{f(x)}\frac{\delta\theta}{\mathrm{J}(\chi,\theta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}.$$

Each of the equations $X=0$, $Y=0$ has mn roots; if $x=x_\mu$, $y=y_{\mu'}$ be a pair which make χ and θ both vanish, these may be called congruous; but if $x=x_\mu$, $y=y_{\mu'}$ do not make χ and θ both vanish then for these

$$\Delta=0,$$

while for congruous values Δ does not vanish.

Moreover, for congruous values of x and y we at once have

$$\frac{dX}{dx}\cdot\frac{dY}{dy}=\Delta.J(\chi,\theta)$$

the congruous values being substituted for the variables, so that

$$\frac{1}{J(\chi,\theta)}=\frac{\Delta}{\frac{dX}{dx}\cdot\frac{dY}{dy}}.$$

Now in (v) the summation is for all the x's and for one of the y's, say $y_{1,\mu}$, which may be regarded in the following way. When the equation

$$\chi(x,y)=0$$

is solved for y in terms of x, there will be n roots; take one of these and denote it by y_1, which is therefore a function of x. Substitute in turn $x_1, x_2, \ldots, x_{mn}$; then we obtain for y_1 a series of values, but all derived from the single root of χ. Thus

$$\Sigma\frac{T}{x-\alpha}\,\frac{\delta\theta}{J(\chi,\theta)}=\Sigma\frac{T_\mu}{x_\mu-\alpha}\,\delta\theta_\mu\,\frac{\Delta_\mu}{\frac{dX}{dx}\cdot\frac{dY}{dy}}$$

since we have $x=x_\mu$, $y=y_{1,\mu}$ $(\mu=1, 2, \ldots, mn)$ as the mn congruous roots. Moreover for roots other than these

$$\Delta=0$$

so that we may add on a number of vanishing terms to the right-hand side, and the removal of the restriction now gives

$$\Sigma\frac{T_\mu}{x_\mu-\alpha}\,\delta\theta_\mu\,\frac{\Delta_\mu}{\frac{dX}{dx_\mu}\,\frac{dY}{dy_{\mu'}}}$$

(where $\mu'=1,\mu$ or $2,\mu$ or $\ldots$ or n,μ).

Moreover from (iv)

$$\Delta\chi=DX-BY$$

and therefore

$$\Delta_\mu\frac{\delta\chi}{\delta y_\mu}=-B_{\mu,\mu'}\frac{dY}{dy_{\mu'}}$$

the term corresponding to a differential of X disappearing and the others vanishing in virtue of the values assigned to the two variables; thus our expression becomes

$$-\Sigma \frac{T_\mu}{x_\mu - \alpha} \delta\theta_\mu \frac{B_{\mu\iota'}}{\frac{\delta\chi}{\delta y_{\mu'}} \cdot \frac{dX}{dx_\mu}}.$$

But the coefficient of $\frac{1}{x}$ in the expansion of $\frac{V}{X}$ in descending powers of x is

$$C_{\frac{1}{x}} \Sigma \frac{V_\iota}{(x - x_\iota) \frac{dX}{dx_\iota}}$$

$$= \Sigma \frac{V_\iota}{\frac{dX}{dx_\iota}}$$

and therefore the foregoing

$$= -C_{\frac{1}{x}} \Sigma \frac{T}{x - \alpha} \delta\theta \frac{B_{\mu'}}{X \frac{\delta\chi}{\delta y_{\mu'}}}$$

the Σ referring to the n values of y obtained from the equation (i), and the expansion being in the factors of X alone. But since we are substituting for y from (i) we have χ always zero in this, and therefore

$$X = B_{\mu'}\theta.$$

Taking now into account the expansion for the factor $\frac{1}{x - \alpha}$, we have finally

$$\Sigma \frac{T}{x - \alpha} \frac{\delta\theta}{J(\chi, \theta)} = -C_{\frac{1}{x}} \Sigma \frac{T}{x - \alpha} \frac{\delta\theta}{\theta} \frac{1}{\frac{\delta\chi}{\delta y}} + \Sigma \left[\frac{T}{\frac{\delta\chi}{\delta y}} \frac{\delta\theta}{\theta} \right]_{x = \alpha}$$

the summation in each of the terms on the right-hand side being for the n values of y. Now on the introduction of BOOLE's symbol Θ (*cf.* Phil. Trans., 1857, p. 751), the right-hand side is merely the definition of

$$\Theta \left[\frac{1}{x - \alpha} \right] \Sigma \frac{T}{\frac{\delta\chi}{\delta y}} \frac{\delta\theta}{\theta}.$$

Let $x = \alpha$ be a root of $f(x) = 0$, and in (v) expand $[f(x)]^{-1}$ in a series of partial fractions corresponding to the roots; then expressions of the form $\frac{T}{x - \alpha} \frac{\delta\theta}{J(\chi, \theta)}$ are obtained. Moreover, from the nature of the preceding fractions and the definition of the symbol Θ in connexion with them, it is obviously a distributive symbol; thus we have

$$\Sigma\frac{\mathrm{T}}{f(x)}\frac{dx}{\frac{\partial\chi}{\partial y}}=\Sigma\frac{\mathrm{T}}{f(x)}\frac{\delta\theta}{\mathrm{J}(\chi,\theta)}$$

$$=\Theta\left[\frac{1}{f(x)}\right]\Sigma\frac{\mathrm{T}}{\frac{\partial\chi}{\partial y}}\frac{\delta\theta}{\theta}$$

where the summation on the left-hand side is over the roots of the equation

$$\mathrm{X}=0$$

while on the right-hand side it is over the n roots y of the equation $\chi(x, y)=0$. The variables on the right-hand side being the arbitrary constants in θ which occur only in the factor $\frac{\delta\theta}{\theta}$, we may integrate, and we have as the result

$$\sum_{\mu=1}^{\mu=mn}\int^{x_\mu}\frac{\mathrm{T}}{f(x)}\frac{dx}{\frac{\partial\chi}{\partial y}}=\Theta\left[\frac{1}{f(x)}\right]\sum_{y=y_1}^{y=y_n}\left\{\frac{\mathrm{T}}{\frac{\partial\chi}{\partial y}}\log\theta\right\}+\mathrm{C}$$

agreeing with the form given in Professor ROWE's memoir (Phil. Trans. 1881, p. 721).

7. In the generalisation of the theorem we shall consider only two dependent variables y, z and one independent variable x; it will be seen that the work would apply, *mutatis mutandis*, to $k-1$ dependent variables and one independent. Let the variables y and z be given as functions of x by the equations

$$\mathrm{F}_m\ (x, y, z)=0$$
$$\mathrm{F}_n\ (x, y, z)=0$$

of the degrees m and n respectively. Let

$$\mathrm{F}_p\ (x, y, z)$$

be a function of x, y and z the coefficients of y and z in which are functions of x with any number of arbitrary constants; so that as in the simple case when z, x and y vary the constants also vary. Using the same notation as before we have

$$\frac{\partial\mathrm{F}_m}{\partial x}dx+\frac{\partial\mathrm{F}_m}{\partial y}dy+\frac{\partial\mathrm{F}_m}{\partial z}dz=0,$$

$$\frac{\partial\mathrm{F}_n}{\partial x}dx+\frac{\partial\mathrm{F}_n}{\partial y}dy+\frac{\partial\mathrm{F}_n}{\partial z}dz=0.$$

Therefore

$$\frac{dz}{\frac{\partial(\mathrm{F}_m,\mathrm{F}_n)}{\partial(x,y)}}=\frac{dx}{\frac{\partial(\mathrm{F}_m,\mathrm{F}_n)}{\partial(y,z)}}=\frac{dy}{\frac{\partial(\mathrm{F}_m,\mathrm{F}_n)}{\partial(z,x)}}=dw$$

and

$$\frac{\delta F_p}{\delta z}dz+\frac{\delta F_p}{\delta x}dx+\frac{\delta F_p}{\delta y}dy+\delta F_p=0$$

so that

$$dw.J\left(\frac{F_m, F_n, F_p}{x, y, z}\right)+\delta F_p=0$$

and therefore

$$-\Sigma T dw=\Sigma\frac{T}{J\left(\frac{F_m, F_n, F_p}{x, y, z}\right)}\delta F_p \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i).}$$

Let Z, X and Y be the eliminants of F_m, F_n and F_p respectively with regard to x and y, y and z, z and x; then as explained in § 3 we may write

$$X=A_mF_m+A_nF_n+A_pF_p$$
$$Y=B_mF_m+B_nF_n+B_pF_p$$
$$Z=C_mF_m+C_nF_n+C_pF_p.$$

Each of the equations $z=0$, $x=0$, $y=0$, has mnp solutions; let those values (mnp in all) which make F_m, F_n and F_p vanish simultaneously be called congruous.

Write

$$\Delta=\begin{vmatrix} A_m & A_n & A_p \\ B_m & B_n & B_p \\ C_m & C_n & C_p \end{vmatrix}$$

so that for non-congruous values Δ is zero.

Now whatever be the value of T it can be put into the form

$$\frac{\Psi(x, y, z)}{\Phi(x, y, z)}$$

where Φ and Ψ are rational integral algebraical functions of x, y and z; and this can be expressed as

$$\frac{U}{f(x)}$$

where U is an integral function of x, y and z and $f(x)$ is a function of x only. For it is

$$\frac{\Psi(x, y_1, z_1)\prod\limits_{\mu=2}^{\mu=mn}\Phi(x, y_\mu, z_\mu)}{\prod\limits_{\mu=1}^{\mu=mn}\Phi(x, y_\mu, z_\mu)}$$

where y_μ, z_μ are a pair of values of y and z which satisfy the equations

$$F_m=0, \quad F_n=0.$$

Now the denominator will consist of symmetric functions of the y's and z's, the coefficients in its different terms involving x. These can be expressed in terms of x alone,[*] while the new term in the numerator can be expressed in terms of symmetric functions of the y's and z's and of y_1 and z_1, and thus T is reduced to the form

$$\frac{U}{f(x)}$$

and therefore

$$-\Sigma T dw = \Sigma \frac{U}{f(x)} \frac{\delta F_p}{J\left(\frac{F_m, F_n, F_p}{x, y, z}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

Let $x=\alpha$ be any root of $f(x)=0$; then, as before, we consider

$$-\Sigma \frac{U}{x-\alpha} dw,$$

that is,

$$\Sigma \frac{U}{x-\alpha} \frac{\delta F_p}{J},$$

the summation being for the mnp values of x; and a definite value of y and one of z are to be substituted in terms of x before the summation is effected.

Having these definite values of y and z (obtained from $F_m=0$, $F_n=0$) if in them we substitute in turn the mnp values of x, we shall have mnp congruous values and therefore all the congruous values. For these, as we easily see,

$$\frac{1}{J\left(\frac{F_m, F_n, F_n}{x, y, z}\right)} = \frac{\Delta}{\frac{dZ}{dz} \frac{dX}{dx} \frac{dY}{dy}}$$

and therefore

$$-\Sigma \frac{U}{x-\alpha} dw = \Sigma \frac{U}{x-\alpha} \delta F_p \frac{\Delta}{\frac{dZ}{dz} \frac{dX}{dx} \frac{dY}{dy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

the summation on each side being the same. But for all values not included in this summation we have $\Delta=0$, and therefore the restrictions on the right-hand side may be removed without altering its value, and we shall consider the summation to extend over all the roots of $F_m=0$, $F_n=0$ considered as equations in y and z and over all the roots x.

Let α denote the minor of A, β that of B, γ that of C (in each case with the same suffix) in Δ. Then we have

$$\alpha_m X + \beta_m Y + \gamma_m Z \equiv \Delta F_m$$
$$\alpha_n X + \beta_n Y + \gamma_n Z \equiv \Delta F_n$$
$$\alpha_p X + \beta_p Y + \gamma_p Z \equiv \Delta F_p.$$

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[*] *Cf.* SALMON's 'Higher Algebra,' § 74.

Differentiating the first two of these with respect to y and z separately and then inserting the values of x, y and z as they now occur on the right-hand side of (iii) we have

$$\beta_m \frac{dY}{dy} = \Delta \frac{\delta F_m}{\delta y}$$

$$\gamma_m \frac{dZ}{dz} = \Delta \frac{\delta F_m}{\delta z}$$

$$\beta_n \frac{dY}{dy} = \Delta \frac{\delta F_n}{\delta y}$$

$$\gamma_n \frac{dZ}{dz} = \Delta \frac{\delta F_n}{\delta z}$$

and therefore

$$\Delta^2 \frac{\delta(F_m, F_n)}{\delta(y, z)} = (\beta_m \gamma_n - \beta_n \gamma_m) \frac{dY}{dy} \frac{dZ}{dz}$$

$$= A_p \Delta \frac{dY}{dy} \frac{dZ}{dz}$$

by a known theorem in determinants; thus (iii) becomes

$$-\Sigma \frac{U}{x-\alpha} \delta w = \Sigma \frac{U}{x-\alpha} \delta F_p \frac{A_p}{\frac{dX}{dx}} \frac{1}{\frac{\delta(F_m, F_n)}{\delta(y, z)}}.$$

Now expanding in partial fractions we have

$$\frac{K}{X} = \Sigma \frac{K_\iota}{(x - x_\iota) \frac{dX}{dx_\iota}}$$

and therefore the right-hand side becomes

$$C_{\frac{1}{x}} \Sigma \left[\frac{U}{x-\alpha} \frac{\delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}} \frac{A_p}{X} \right]$$

considered as expanded for the factors of X alone or, including in the expansion the term arising from $\frac{1}{x-\alpha}$, it is equal to

$$C_{\frac{1}{x}} \Sigma \left[\frac{U}{x-\alpha} \frac{\delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}} \frac{A_p}{X} \right] - \Sigma \left[\frac{U \delta F_p}{\frac{\delta(F_m, F_n)}{\delta(y, z)}} \frac{A_p}{X} \right]_{x=\alpha}$$

wherein the Σ implies summation for all values of y and z in terms of x derived from the equations $F_m = 0$ and $F_n = 0$. Since these values are to be substituted we have

$$\begin{aligned}\mathrm{X} &= \mathrm{A}_m\mathrm{F}_m + \mathrm{A}_n\mathrm{F}_n + \mathrm{A}_p\mathrm{F}_p \\ &= \mathrm{A}_p\mathrm{F}_p\end{aligned}$$

and therefore substituting this in the above which will replace the right-hand side of (iii) and inserting the value of dw the equation becomes

$$\begin{aligned}\Sigma\frac{\mathrm{U}}{x-\alpha}\,\frac{dx}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)} &= \Sigma\left[\frac{\mathrm{U}}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}\,\frac{\delta\mathrm{F}_p}{\mathrm{F}_p}\right]_{x=\alpha} - \underset{x}{\mathrm{C}_1}\,\Sigma\left[\frac{\mathrm{U}}{x-\alpha}\,\frac{1}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}\,\frac{\delta\mathrm{F}_p}{\mathrm{F}_p}\right] \\ &= \Theta\left[\frac{1}{x-\alpha}\right].\Sigma\left[\frac{\mathrm{U}}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}\,\frac{\delta\mathrm{F}_p}{\mathrm{F}_p}\right]\end{aligned}$$

by the use of **Boole**'s symbol Θ as before. The summation on the left-hand side is of course over the mnp roots x; on the right-hand side it is over the mn roots y and z in terms of x of the equations $\mathrm{F}_m=0$ and $\mathrm{F}_n=0$. We may obviously integrate as before; and using the distributive property of Θ we obtain as our result

$$\sum_{\mu=1}^{\mu=mnp}\int^{x_\mu}\frac{\mathrm{U}}{f(x)}\,\frac{dx}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)} - \mathrm{C} = \Theta\left[\frac{1}{f(x)}\right].\sum_{\mu=1}^{\mu=mn}\left\{\frac{\mathrm{U}}{\mathrm{J}\left(\frac{\mathrm{F}_m,\ \mathrm{F}_n}{y,\ z}\right)}\log\mathrm{F}_p\right\} \quad . \quad . \quad . \quad \text{(iv).}$$

8. The general theorem will proceed on lines not widely different from the above, and may be enunciated as follows. Let

$$\begin{aligned}&\mathrm{F}_1\ \ (x_1, x_2, \ldots, x_r) = 0 \\ &\mathrm{F}_2\ \ (x_1, x_2, \ \ldots, x_r) = 0 \\ &\qquad \cdots\cdots \\ &\mathrm{F}_{r-1}(x_1, x_2, \ldots, x_r) = 0\end{aligned}$$

be $r-1$ equations, of degrees $m_1, m_2, \ldots, m_{r-1}$ respectively, giving $x_2, \ldots, x_r$ in terms of x_1; and let

$$\mathrm{F}_r(x_2, x_3, \ldots, x_r)$$

be a function of these dependent variables, the coefficients of which are functions of x_1 containing any number of arbitrary constants. Form the eliminant E of all the F's so that we shall obtain the set of roots x_1 by equating E to zero; and denote by U any algebraical rational integral function of $x_1, x_2, \ldots x_r$. Then

$$\Sigma\int\frac{\mathrm{U}}{f(x_1)}\,\frac{dx_1}{\mathrm{J}\left(\frac{\mathrm{F}_1,\ \mathrm{F}_2,\ \ldots,\ \mathrm{F}_{r-1}}{x_2,\ x_3,\ \ldots,\ x_r}\right)} = \Theta\left[\frac{1}{f(x_1)}\right].\Sigma\left\{\frac{\mathrm{U}\log\mathrm{F}_r}{\mathrm{J}\left(\frac{\mathrm{F}_1,\ \mathrm{F}_2,\ \ldots,\ \mathrm{F}_{r-1}}{x_2,\ x_3,\ \ldots,\ x_r}\right)}\right\} + \mathrm{A}$$

the summation on the left-hand side being taken over all the roots of the equation $E=0$, which are assumed as the upper limits of the integrals, while on the right-hand side the summation is over all the roots of $F_1=0$, $F_2=0, \ldots, F_{r-1}=0$ considered as $r-1$ simultaneous equations giving $x_2, x_3, \ldots, x_r$ in terms of x_1.

9. Returning now to (iv) the conditions that the right-hand side should reduce to a constant are :—

1°: That $f(x)\equiv 1$, or be a factor of U;

and 2°:
$$\underset{x}{\mathrm{C}_1}\Sigma\frac{\mathrm{U}}{\mathrm{J}}\log\theta=0$$

which will be satisfied by
$$\underset{x}{\mathrm{C}_1}\frac{\mathrm{U}}{\mathrm{J}}=0.$$

Now J is of the order $m+n-2$ in x, and therefore the order of U may not be greater than $m+n-4$. In this case the number of terms in it will be

$$\frac{m+n-1\ m+n-2.m+n-3}{1\ 2\ 3}.$$

But if the integrals $\int\frac{x^k}{\mathrm{J}}dx$ be formed they are not all independent for

$$\int\frac{\mathrm{V}_n\mathrm{F}_m}{\mathrm{J}}dx=0$$

$$\int\frac{\mathrm{V}_m\mathrm{F}_n}{\mathrm{J}}dx=0$$

where V_m and V_n are arbitrary functions of the orders $m-4$ and $n-4$ respectively, and contain

$$\frac{m-1\ m-2.m-3}{1.2.3}\text{ and }\frac{n-1\ n-2\ n-3}{1.2.3}$$

terms. Hence the number of independent integrals in the case when the right-hand side reduces to a constant or to zero is

$$\frac{m+n-1\ m+n-2.m+n-3}{1.2.3}-\frac{m-1.m-2.m-3}{1.2\ 3}-\frac{n-1.n-2\ n-3}{1.2.3}=\tfrac{1}{2}mn(m+n-4)+1$$

This assumes that the surfaces F_m and F_n are the most general of the degrees m and n respectively and so possess no special singularities.

10. ABEL'S theorem in the more simple case applies to the intersection of plane curves. There is a fixed curve given by

$$\chi(x, y)=0$$

and integrals connected with this curve are taken, the upper limits assigned being the abscissæ of its points of intersection with another curve, the equation to which

$$\theta(x, y)=0$$

contains a number of variable parameters and therefore represents a variable curve.

But the more general form of the theorem extends the application to curves in space. We take the curve which is the intersection of two surfaces

$$\mathrm{F}_m(x, y, z)=0$$
$$\mathrm{F}_n(x, y, z)=0$$

(and which will, as a rule, be a tortuous curve), and forming the corresponding integrals we assign as the upper limits of these the ordinates x of the points of intersection of this tortuous curve with a surface the equation to which

$$\mathrm{F}_p(x, y, z)=0$$

containing a number of variable parameters represents a variable surface.

The discussion of this geometrical interpretation and of the deductions to which it leads has been carried out in a memoir by CLEBSCH ('Crelle,' t. lxiii., p. 189, 1863), wherein he proceeds from the theorems which are the forms of (iv) and (v) when the right-hand sides are zero. Example I. which follows was suggested by an analogous geometrical illustration which Professor CAYLEY gave in one of his lectures at Cambridge in the Michaelmas Term, 1881, wherein he pointed out how to obtain $\mathrm{sn}(u+v+w)$ from the analytical expression for the co-planarity of the four points of intersection of an arbitrary plane (corresponding to $\mathrm{F}_p=0$) with a fixed tortuous curve in space which was the intersection of a circular cylinder and an elliptic cylinder respectively corresponding to $\mathrm{F}_m=0$ and $\mathrm{F}_n=0$.

We now proceed to consider two examples of (iv).

11. *Example I.*

Let
$$\mathrm{F}_m\equiv y^2-(1-x^2)=0$$
$$\mathrm{F}_n\equiv z^2-(1-k^2x^2)=0$$
$$\mathrm{F}_p\equiv \mathrm{A}x+\mathrm{B}y+\mathrm{C}z-1.$$

The eliminant X is obviously

$$\mathrm{X}=\Pi\{\mathrm{A}x-1\pm\mathrm{B}(1-x^2)^{\frac{1}{2}}\pm\mathrm{C}(1-k^2x^2)^{\frac{1}{2}}\}$$

Π denoting the product of the four expressions which the above includes owing to the two double signs. It is evidently of the fourth degree in x; let the roots be x_1, x_2, x_3, x_4. As there are three arbitrary constants there will be one relation between these four roots, and this can be exhibited in the form

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

If we choose $\mathrm{A}=0$ and one of the two, B and C, to be unity and the other zero, $\mathrm{X}=0$ reduces to

$$x^4=0$$

and we may therefore take zero as the lower limits of all our integrals.

Let

$$y_1=+(1-x^2)^{\frac{1}{2}}$$
$$z_1=+(1-k^2x^2)^{\frac{1}{2}}$$

then $\pm y_1$ are the roots of F_m, $\pm z_1$ those of F_n. We have

$$\mathrm{J}=4y_1z_1=4(1-x^2.1-k^2x^2)^{\frac{1}{2}}$$

and therefore by our formula (iv)

$$\sum_{\mu=1}^{\mu=4}\int_0^{x_\mu}\frac{\mathrm{U}dx}{f(x)\sqrt{(1-x^2)(1-k^2x^2)}}=\Theta\left[\frac{1}{f(x)}\right]\Sigma\left\{\frac{\mathrm{U}}{yz}\log\left(1-\mathrm{A}x-\mathrm{B}y-\mathrm{C}z\right)\right\}$$

in which the Σ on the right-hand side implies summation for the expressions obtained by the substitutions

$$\begin{aligned} y&=y_1 \text{ and } z=z_1, \\ y&=-y_1 \quad ,, \quad z=z_1, \\ y&=y_1 \quad ,, \quad z=-z_1, \\ y&=-y_1 \quad ,, \quad z=-z_1. \end{aligned}$$

(i.) Let $f(x)\equiv 1$, $\mathrm{U}=1$; then the right-hand side vanishes and we have

$$u_1+u_2+u_3+u_4=0$$

where

$$x=\mathrm{sn}\,u.$$

Thus the preceding determinantal relation will give $\mathrm{sn}(u_1+u_2+u_3)$, which is $-x_4$, in terms of the elliptic functions of u_1, u_2, u_3.

(ii.) Let $f(x)\equiv 1$, $\mathrm{U}=z^2=1-k^2x^2$; then we have

$$\begin{aligned} &\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)+\mathrm{E}(u_4) \\ &=-\mathrm{C}_{\frac{1}{x}}\,\Sigma\left\{\frac{z}{y}\log\left(1-\mathrm{A}x-\mathrm{B}y-\mathrm{C}z\right)\right\} \\ &=-\mathrm{C}_{\frac{1}{x}}\left[\frac{z_1}{y_1}\log\left(\frac{1-\mathrm{A}x-\mathrm{C}z_1-\mathrm{B}y_1}{1-\mathrm{A}x-\mathrm{C}z_1+\mathrm{B}y_1}\right)+\frac{z_1}{y_1}\log\left(\frac{1-\mathrm{A}x+\mathrm{C}z_1+\mathrm{B}y_1}{1-\mathrm{A}x+\mathrm{C}z_1-\mathrm{B}y_1}\right)\right]. \end{aligned}$$

Expanding the logarithms on the right-hand side the n^{th} term gives

$$\frac{2}{2n-1}\,C_{\frac{1}{x}}\left[z_1\frac{B^{2n-1}y_1^{\,2n-2}}{(1-Ax-Cz_1)^{2n-1}}-z_1\frac{B^{2n-1}y_1^{\,2n-2}}{(1-Ax+Cz_1)^{2n-1}}\right]$$

$$=\frac{2B^{2n-1}}{2n-1}\,C_{\frac{1}{x}}\left[y_1^{\,2n-2}\frac{z_1(1-Ax+Cz_1)^{2n-1}-z_1(1-Ax-Cz_1)^{2n-1}}{\{(1-Ax)^2-C^2z_1^{\,2}\}^{2n-1}}\right]$$

$$=\frac{4B^{2n-1}C}{2n-1}\,C_{\frac{1}{x}}\left[(1-x^2)^{n-1}(1-k^2x^2)\right.$$

$$\left.\frac{\left\{(2n-1)(1-Ax)^{2n-2}+\frac{2n-1.2n-2.2n-3}{1\,2\,3}(1-Ax)^{2n-4}C^2(1-k^2x^2)+\ldots\right\}}{(A^2+k^2C^2)^{2n-1}x^{4n-2}\left\{1-\frac{2A}{A^2+k^2C^2}\frac{1}{x}+\ \ldots\right\}^{2n-1}}\right]$$

$$=\frac{(-1)^n4k^2B^{2n-1}C}{(2n-1)(A^2+k^2C^2)^{2n-1}}\,C_{\frac{1}{x}}\left[\left\{1+\frac{2(2n-1)A}{A^2+k^2C^2}\cdot\frac{1}{x}-\ldots\right\}\right.$$

$$\left\{(2n-1)A^{2n-2}-(2n-1)(2n-2)A^{2n-3}\frac{1}{x}-k^2C^2\left(\frac{2n-1.2n-2\,2n-3}{1\,2.3}A^{2n-4}\right.\right.$$

$$\left.\left.\left.-\frac{2n-1\,.\,2n-4}{1.2.3}A^{2n-5}\frac{1}{x}\right)+\ldots\right\}\right]$$

$$=\frac{(-1)^n4k^2B^{2n-1}C}{(2n-1)(A^2+k^2C^2)^{2n-1}}\left[-(2n-1)\left\{(2n-2)A^{2n-3}-k^2C^2\frac{2n-2.2n-3\,2n-4}{1.2.3}A^{2n-5}+\ldots\right\}\right.$$

$$\left.+\frac{2\,2n-1\,A}{A^2+k^2C^2}\left\{(2n-1)A^{2n-2}-k^2C^2\frac{2n-1\,2n-2.2n-3}{1.2.3}A^{2n-4}+\ldots\right\}\right]$$

$$=\frac{(-1)^n4k^2B^{2n-1}C}{(A^2+k^2C^2)^{2n}}\,\frac{(A+ikC)^{2n}-(A-ikC)^{2n}}{2ikC}$$ after a slight reduction and writing $i=\sqrt{-1}$

$$=\frac{(-1)^n2kB^{2n-1}}{i}\left[\frac{1}{(A-ikC)^{2n}}-\frac{1}{(A+ikC)^{2n}}\right].$$

Hence the whole coefficient as derived from all the terms in the expansion is

$$=\frac{2k}{i}\left[-\frac{B}{(A-ikC)^2}+\frac{B^3}{(A-ikC)^4}-\ldots+\frac{B}{(A+ikC)^2}-\frac{B^3}{(A+ikC)^4}+\ldots\right]$$

$$=-\frac{2k}{i}\left[\frac{B}{B^2+(A-ikC)^2}-\frac{B}{B^2+(A+ikC)^2}\right]$$

$$=-\frac{8k^2ABC}{(A^2+B^2-k^2C^2)^2+4k^2A^2C^2}$$

and this is the value of

$$E(u_1)+E(u_2)+E(u_3)-E(u_1+u_2+u_3).$$

Writing s, c, d respectively for sn$u$, cn$u$, dn$u$, the values of A, B and C are given by the equations

$$\begin{aligned} \mathrm{A}s_1+\mathrm{B}c_1+\mathrm{C}d_1&=1 \\ \mathrm{A}s_2+\mathrm{B}c_2+\mathrm{C}d_2&=1 \\ \mathrm{A}s_3+\mathrm{B}c_3+\mathrm{C}d_3&=1 \end{aligned}$$

or writing

$$\begin{aligned} c_2d_3-c_3d_2+\qquad\quad &=\sigma \\ d_2s_3-d_3s_2+\ .\quad .\ &=\kappa \\ s_2c_3-s_3c_2+\ .\ .\quad &=\delta \end{aligned}$$

$$\begin{vmatrix} s_1 & c_1 & d_1 \\ s_2 & c_2 & d_2 \\ s_3 & c_3 & d_3 \end{vmatrix}=\Delta$$

we have

$$\Delta\mathrm{A}=\sigma \qquad \Delta\mathrm{B}=\kappa \qquad \Delta\mathrm{C}=\delta$$

and then

$$\begin{aligned} &\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-\mathrm{E}(u_1+u_2+u_3) \\ &\quad=-\frac{8k^2\sigma\kappa\delta\Delta}{(\sigma^2+\kappa^2-k^2\delta^2)^2+4k^2\kappa^2\delta^2}. \end{aligned}$$

As a verification of our formula assume

$$u_1=u_2=u_3$$

so that

$$x_1=x_2=x_3.$$

Then since the equation

$$\mathrm{A}s+\mathrm{B}c+\mathrm{C}d=1$$

has three equal roots, the values of A, B and C for our case will obviously be given by

$$\begin{aligned} \mathrm{A}s_1+\mathrm{B}c_1+\mathrm{C}d_1\ \ &=1 \\ \mathrm{A}\quad-\mathrm{B}\frac{s_1}{c_1}-\mathrm{C}k^2\frac{s_1}{d_1}&=0 \\ \mathrm{B}\frac{1}{c_1{}^3}+\mathrm{C}k^2\frac{1}{d_1{}^3}&=0\,; \end{aligned}$$

for if we write $s_1+\xi$ for s_1 in the first, the coefficient of ξ must vanish, which condition gives the second equation; and similarly for the third. These last two equations give

$$\frac{\mathrm{A}}{k^2k'^2s_1{}^3}=\frac{\mathrm{B}}{-k^2c_1{}^3}=\frac{\mathrm{C}}{d_1{}^3}=\mu$$

and substituting in the first we find

$$\mu=\frac{1}{k'^2}.$$

From these we have at once

$$\frac{k'^4}{k^2}(A^2+B^2-k^2C^2)=k^2k'^4s^6-d^6+k^2c^6 \quad \text{(dropping subscripts)}$$

$$=-k'^2(1-3k^2s^4+2k^4s^6)$$

$$\frac{k'^2}{k^2}AC=s^3d^3.$$

Hence

$$\frac{k'^4}{k^4}\{(A^2+B^2-k^2C^2)^2+4k^2A^2C^2\}=(1-3k^2s^4+2k^4s^6)^2+4k^2s^6(1-k^2s^2)^3$$

$$=1-6k^2s^4+4(k^2+k^4)s^6-3k^2s^8$$

and we therefore have to verify that

$$3E(u)-E(3u)=\frac{8k^2s^3c^3d^3}{1-6k^2s^4+4(k^2+k^4)s^6-3k^4s^8}.$$

Now the ordinary addition formula for E is

$$E(u)+E(v)-E(u+v)=k^2\operatorname{sn}u\operatorname{sn}v\operatorname{sn}(u+v)$$

so that

$$E(u)+E(2u)-E(3u)=k^2\operatorname{sn}u\operatorname{sn}2u\operatorname{sn}3u$$

$$2E(u)-E(2u)=k^2\operatorname{sn}^2u\operatorname{sn}2u$$

and hence

$$3E(u)-E(3u)=k^2\operatorname{sn}u\operatorname{sn}2u(\operatorname{sn}u+\operatorname{sn}3u).$$

But

$$\operatorname{sn}3u=\frac{3s-(4+4k^2)s^3+6k^2s^5-k^4s^9}{1-6k^2s^4+4(k^2+k^4)s^6-3k^4s^8}$$

or writing D for the denominator

$$D(\operatorname{sn}u+\operatorname{sn}3u)=4s\{1-(1+k^2)s^2+(k^2+k^4)s^4-k^4s^8\}$$

$$=4s(1-k^2s^4)(1-s^2)(1-k^2s^2)$$

$$=4s(1-k^2s^4)c^2d^2.$$

Moreover

$$\operatorname{sn}2u=\frac{2scd}{1-k^2s^4}$$

so that

$$3E(u)-E(3u)=\frac{8k^2s^3c^3d^3}{D}$$

verifying the formula as required.

(iii.) In a similar way if we write

$$\Pi(\lambda,\ u_1)=\int_0^{x_1}\frac{dx}{(1-\lambda^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

(so that $U=1, f(x)=1-\lambda^2x^2$) we shall obtain

$$\Pi(\lambda,\ u_1)+\Pi(\lambda,\ u_2)+\Pi(\lambda,\ u_3)+\Pi(\lambda,\ u_1)$$

$$=-\frac{\lambda}{2\sqrt{(1-\lambda^2)(k^2-\lambda^2)}}$$

$$\cdot\log\left[\frac{\{(A-\lambda)^2+(B\sqrt{1-\lambda^2}+C\sqrt{k^2-\lambda^2})^2\}\{(A+\lambda)^2+(B\sqrt{1-\lambda^2}-C\sqrt{k^2-\lambda^2})^2\}}{\{(A+\lambda)^2+(B\sqrt{1-\lambda^2}+C\sqrt{k^2-\lambda^2})^2\}\{(A-\lambda)^2+(B\sqrt{1-\lambda^2}-C\sqrt{k^2-\lambda^2})^2\}}\right]$$

the values of A, B, C being those which occur in the general case in (ii.).

Let $\lambda=k\,\mathrm{sn}a$ so as to introduce the third elliptic integral in the form used by JACOBI; then

$$\Pi(\lambda,\ u)=u+\frac{\mathrm{sn}a}{\mathrm{cn}a\mathrm{dn}a}\Pi(u,\ a)$$

$$=u+\frac{\lambda}{\sqrt{(1-\lambda^2)(k^2-\lambda^2)}}\Pi(u,\ a)$$

and the form of the theorem is now obviously as follows :—

$$\Pi(u_1,\ a)+\Pi(u_2,\ a)+\Pi(u_3,\ a)+\Pi(u_4,\ a)$$

$$=\tfrac{1}{2}\log\left[\frac{\{(A+ks')^2+(Bd'+Ckc')^2\}\{(A-ks')^2+(Bd'-Ckc')^2\}}{\{(A-ks')^2+(Bd'+Ckc')^2\}\{(A+ks')^2+(Bd'-Ckc')^2\}}\right]$$

where s', c', d' stand respectively for $\mathrm{sn}a$, $\mathrm{cn}a$, $\mathrm{dn}a$.

12. *Example II.*

Take F_m and F_n as in example I., but now let

$$F_p\equiv Azy+(Bx+C)y+(Hx+D)z-Gx^2-Fx-1,$$

in effect the most general quadric relation. The eliminant X will be of the degree 8, and as there are seven arbitrary constants there will be only a single relation between the roots $x_1, x_2, \ldots, x_8$, which can be expressed in the form

$$\begin{vmatrix} c_1d_1 & s_1c_1 & s_1d_1 & s_1^2 & c_1 & d_1 & s_1 & 1 \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ c_8d_8 & s_8c_8 & s_8d_8 & s_8^2 & c_8 & d_8 & s_8 & 1 \end{vmatrix}=0.$$

Moreover, if we choose

$$\mathrm{B}=\mathrm{H}=\mathrm{F}=\mathrm{G}=0$$
$$-\mathrm{A}=\mathrm{C}=\mathrm{D}=1$$

the equation $\mathrm{X}=0$ is of the form

$$x^8=0$$

and we can therefore take zero for the lower limit in all our integrals. Hence we shall have

$$u_1+u_2+\ldots\quad+u_8=0$$

where

$$x=\mathrm{sn}\,u$$

and the above relation will give $\mathrm{sn}(u_1+\ldots+u_7)$ in terms of the elliptic functions of $u_1,\ldots, u_7$.

Let us now find

$$\sum_{\mu=1}^{\mu=8}\int_0^{x_\mu}\sqrt{\frac{1-k^2x^2}{1-x^2}}dx\,;$$

write

$$\mathrm{B}x+\mathrm{C}=w_2;\qquad \mathrm{H}x+\mathrm{D}=w_1;\qquad 1+\mathrm{F}x+\mathrm{G}x^2=w_3;$$

then the right-hand side of the equation is

$$-\mathrm{C}_{\frac{1}{x}}\left[\frac{z}{y}\log\left\{\frac{w_3-zw_1-y(\mathrm{A}z+w_2)}{w_3-zw_1+y(\mathrm{A}z+w_2)}\right\}+\frac{z}{y}\log\left\{\frac{w_3+zw_1+y(\mathrm{A}z-w_2)}{w_3+zw_1-y(\mathrm{A}z-w_2)}\right\}\right].$$

On expansion, the n^{th} term gives

$$\frac{2}{2n-1}\mathrm{C}_{\frac{1}{x}}\left[z\frac{y^{2n-2}(\mathrm{A}z+w_2)^{2n-1}}{(w_3-zw_1)^{2n-1}}+z\frac{y^{2n-2}(\mathrm{A}z-w_2)^{2n-1}}{(w_3+zw_1)^{2n-1}}\right]$$

$$=\frac{2}{2n-1}\mathrm{C}_{\frac{1}{x}}\left[\frac{zy^{2n-2}}{(w_3{}^2-z^2w_1{}^2)^{2n-1}}\{(\mathrm{A}z+w_2)^{2n-1}(w_3+zw_1)^{2n-1}-(w_2-\mathrm{A}z)^{2n-1}(w_3-zw_1)^{2n-1}\}\right].$$

So far as the result is concerned, the expression within the inner bracket is

$$\{w_2w_3-k^2x^2w_1+z(\mathrm{A}w_3+w_1w_2)\}^{2n-1}\{w_2w_3-k^2x^2w_1-z(\mathrm{A}w_3+w_1w_2)\}^{2n-1}$$
$$=2z\{(2n-1)(w_2w_3-k^2x^2w_1)^{2n-2}(\mathrm{A}w_3+w_1w_2)+\ldots\}.$$

Now

$$w_2w_3-k^2x^2w_1=x^3\left[\mathrm{BG}-k^2\mathrm{H}+\frac{1}{x}(\mathrm{CG}+\mathrm{BF}-k^2\mathrm{D})\right]$$
$$=x^3\left(\lambda_1+\frac{1}{x}\lambda_2\right)\text{ say;}$$

$$\begin{aligned} \mathrm{A}w_3+w_1w_2&=x^2\left[\mathrm{AG}+\mathrm{BH}+\frac{1}{x}(\mathrm{AF}+\mathrm{BD}+\mathrm{CH})\right]\\ &=x^2\left(\mu_1+\frac{1}{x}\mu_2\right) \text{ say;}\\ w_3^2-z^2w_1^2&=x^4(\mathrm{G}^2+k^2\mathrm{H}^2)\left\{1+2\frac{\mathrm{FG}+k^2\mathrm{DH}}{\mathrm{G}^2+k^2\mathrm{H}^2}\,\frac{1}{x}\right\}\\ &=x^4\rho_1\left(1+\frac{1}{x}\rho_2\right) \text{ say.}\end{aligned}$$

Thus the n^{th} term gives

$$\begin{aligned}&\frac{4(-1)^nk^2}{2n-1}\,\frac{1}{\rho_1^{2n-1}}\mathrm{C}_{\frac{1}{x}}\left(1-\frac{2n-1}{x}\rho_2\right)\left[(2n-1)\left(\lambda_1^{2n-2}+\frac{2n-2}{x}\lambda_1^{2n-3}\lambda_2\right)\left(\mu_1+\frac{1}{x}\mu_2\right)\right.\\ &\qquad\left.-k^2\frac{2n-1\,2n-2\,2n-3}{1.2.3}\left(\lambda_1^{2n-4}+\frac{2n-4}{x}\lambda_1^{2n-5}\lambda_2\right)\left(\mu_1^3+\frac{3}{x}\mu_1^2\mu_2\right)+\ldots\right]\end{aligned}$$

$$\begin{aligned}&=\frac{-4(-1)^nk^2}{\rho_1^{2n-1}}\rho_2\left\{(2n-1)\lambda_1^{2n-2}\mu_1-k^2\frac{2n-1.2n-2\,2n-3}{1\,2.3}\lambda_1^{2n-4}\mu_1^3+\ldots\right\}\\ &+\frac{4(-1)^nk^2}{\rho_1^{2n-1}}\left\{\lambda_1^{2n-2}\mu_2-k^2\frac{2n-2\,2n-3}{1\,2}\lambda_1^{2n-4}\mu_1^2\mu_2+\ldots\right.\\ &\qquad\left.+(2n-2)\lambda_1^{2n-3}\lambda_2\mu_1-k^2\frac{2n-2.2n-3\,2n-4}{1.2.3}\lambda_1^{2n-5}\lambda_2\mu_1^3+\ldots\right\}\end{aligned}$$

$$\begin{aligned}&=\frac{4(-1)^nk^2}{\rho_1^{2n-1}}\left[-\frac{\rho_2}{2ik}\{(\lambda_1+ik\mu_1)^{2n-1}-(\lambda_1-ik\mu_1)^{2n-1}\}+\frac{\mu_2}{2}\{(\lambda_1+ik\mu_1)^{2n-2}+(\lambda_1-ik\mu_1)^{2n-2}\}\right.\\ &\qquad\left.+\frac{\lambda_2}{2ik}\{(\lambda_1+ik\mu_1)^{2n-2}-(\lambda_1-ik\mu_1)^{2n-2}\}\right]\end{aligned}$$

i denoting $\sqrt{-1}$. Hence summing up for all the terms and reducing we have the whole coefficient equal to

$$-\frac{4k^2\{(\rho_1^2-k^2\mu_1^2)(\mu_2\rho_1-\rho_1\rho_2\mu_1)+\lambda_1^2(\mu_2\rho_1+\rho_1\rho_2\mu_1)-2\lambda_1\lambda_2\mu_1\rho_1\}}{(\rho_1^2+\lambda_1^2-k^2\mu_1^2)^2+4k^2\lambda_1^2\mu_1^2}.$$

Now

$$\begin{aligned}\rho_1&=\mathrm{G}^2+k^2\mathrm{H}^2 & \mu_1&=\mathrm{AG}+\mathrm{BH}\\ \rho_1\rho_2&=2(\mathrm{FG}+k^2\mathrm{DH}), & \mu_2&=\mathrm{AF}+\mathrm{BD}+\mathrm{CH}\\ \lambda_1&=\mathrm{BG}-k^2\mathrm{H}, & \lambda_2&=\mathrm{CG}+\mathrm{BF}-k^2\mathrm{D}\end{aligned}$$

and the values of A, B, C, D, H, F, G are determined by the seven equations

$$\mathrm{A}c_\mu d_\mu+\mathrm{B}s_\mu c_\mu+\mathrm{H}s_\mu d_\mu-\mathrm{G}s_\mu^2+\mathrm{C}c_\mu+\mathrm{D}d_\mu-\mathrm{F}s_\mu=1$$

($\mu=1, 2, \ldots, 7$); and therefore the above is the value of

$$\sum_{\mu=1}^{\mu=7} \mathrm{E}(u_\mu)-\mathrm{E}\left(\sum_{\mu=1}^{\mu=7} u_\mu\right)$$

expressed in terms of the functions of the u's.

The evaluation of the corresponding expression for the sum of the third elliptic integrals presents no difficulty.

SECTION II.

Abelian functions, after WEIERSTRASS.

13. The theory of these functions is detailed in a paper by WEIERSTRASS in CRELLE'S Journal, t. lii., but such formulæ as may be necessary in what follows will be proved. Let

$$\left.\begin{aligned} y^2-\mathrm{P}(x)&\equiv y^2-(x-a_1)(x-a_2)\ldots(x-a_\rho)=0 \\ z^2-\mathrm{Q}(x)&\equiv z^2-(x-a_{\rho+1})(x-a_{\rho+2})\ldots(x-a_{2\rho+1})=0 \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

and

$$\theta\equiv\mathrm{M}y+\mathrm{N}z \quad \ldots\ldots\ldots\ldots \quad (2)$$

where M is of the degree ρ in x, N of $\rho-1$; say

$$\left.\begin{aligned} \mathrm{M}&\equiv x^\rho+\mathrm{M}_1x^{\rho-1}+\ldots+\mathrm{M}_{\rho-1}x+\mathrm{M}_\rho \\ \mathrm{N}&\equiv \phantom{x^\rho+{}}\mathrm{N}_1x^{\rho-1}+\ldots+\mathrm{N}_{\rho-1}x+\mathrm{N}_\rho \end{aligned}\right\} \quad \ldots\ldots \quad (3).$$

Then the equation for the roots x being

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2=0$$

is of the degree 3ρ and involves 2ρ arbitrary constants; thus there must be ρ relations among the roots. Let these roots be denoted by $x_1, x_2, \ldots, x_\rho$; $\xi_1, \xi_2, \ldots, \xi_\rho$; $p_1, p_2, \ldots, p_\rho$; so that we may consider the ρ p's as given in terms of the x's and ξ's by the ρ relations which might be exhibited in a determinantal form. Write

$$\mathrm{R}(x)=\mathrm{P}(x)\mathrm{Q}(x)$$

and let

$$u_\mu=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{x_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}} \quad \ldots\ldots\ldots\ldots \quad (4)$$

in which μ has in succession the values 1, 2, . . . , ρ as also in

$$v_\mu=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{\xi_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}} \quad \ldots\ldots\ldots\ldots \quad (5)$$

and in

$$w_{\mu}=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_{\lambda}}^{p_{\lambda}}\frac{\mathrm{P}(x)dx}{(x-a_{\mu})\sqrt{\mathrm{R}(x)}}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6).$$

Now the constants in M and N can be so chosen that the roots of the equation

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2=0$$

shall be $a_1, a_2, \ldots, a_\rho$ each occurring thrice; for the conditions are that any one, say a_λ, shall satisfy the three equations

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2=0$$

$$\mathrm{M}^2y\frac{dy}{dx}+y^2\mathrm{M}\frac{d\mathrm{M}}{dx}-\mathrm{N}^2z\frac{dz}{dx}-z^2\mathrm{N}\frac{d\mathrm{N}}{dx}=0$$

$$\mathrm{MT}+y^2\left(\frac{d\mathrm{M}}{dx}\right)^2-\mathrm{NU}-z^2\left(\frac{d\mathrm{N}}{dx}\right)^2=0$$

where T, U are functions of x. In order that a_λ may satisfy the first of these

$$\mathrm{N}=0$$

when $x=a_\lambda$; that it may satisfy the second the additional condition is that

$$\mathrm{M}=0$$

and therefore M will contain $x-a_\lambda$ as a factor. Moreover, y^2 contains $x-a_\lambda$ as a factor and therefore in M^2y^2 we shall have $x-a_\lambda$ raised to the third power; and therefore if we expand N in ascending powers of $x-a_\lambda$ the first term is of the order

$$(x-a_\lambda)^{\frac{3}{2}}.$$

Hence $\left(\frac{d\mathrm{N}}{dx}\right)^2$ has $\mathrm{A}(x-a_\lambda)$ as its first term and so vanishes when $x=a_\lambda$. But this is the additional condition that $x=a_\lambda$ should satisfy the third equation, and being satisfied it proves that a_λ may occur as a triple root provided

$$\mathrm{M}=0 \qquad \mathrm{N}=0$$

when $x=a_\lambda$. That is to say, two conditions are necessary for each root, or 2ρ in all; but as there are 2ρ disposable constants these can all be satisfied and so the truth of the proposition is established. But as N is only of the degree $\rho-1$ in x, while it has to vanish for ρ values, it must be identically zero; and we choose $\mathrm{M}=y^2$, so that the equation is $\mathrm{M}^2y^2=0$ which is obvious beforehand.

14. Applying now the general theorem from § 6 we have

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\int^{x_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_{\mu})\sqrt{\mathrm{R}(x)}}+\int^{\xi_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_{\mu})\sqrt{\mathrm{R}(x)}}+\int^{p_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_{\mu})\sqrt{\mathrm{R}(x)}}\right\}=\text{constant}$$

since $\frac{\mathrm{P}(x)}{x-a_\mu}$ is an integral function of x, and when $\frac{\mathrm{P}(x)}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}$ is expanded in descending powers of x the highest index of x is $-\frac{3}{2}$. We have shown that $a_1, a_2, \ldots, a_\lambda$ may be regarded as triple roots of the equation for the roots, and thus we may take as the constant

$$3\sum_{\lambda=1}^{\lambda=\rho}\int^{a_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}.$$

Hence

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\int_{a_\lambda}^{x_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}+\int_{a_\lambda}^{\xi_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}+\int_{a_\lambda}^{p_\lambda}\frac{\mathrm{P}(x)dx}{(x-a_\mu)\sqrt{\mathrm{R}(x)}}\right\}=0$$

or

$$u_\mu+v_\mu+w_\mu=0 \quad \ldots \ldots \ldots \quad (7).$$

Now, by WEIERSTRASS's theory, given values of $u_1, u_2, \ldots, u_\rho$ imply unique values of $x_1, x_2, \ldots, x_\rho$ which are, in fact, the roots of an equation of the ρ^{th} degree whose coefficients are single-valued functions of $u_1, u_2, \ldots, u_\rho$. Every symmetrical function of $x_1, \ldots, x_\rho$ can therefore be expressed as a function $u_1, u_2, \ldots, u_\rho$, but in particular

$$(a_t-x_1)(a_t-x_2)\ldots(a_t-x_\rho)$$

(t being any of the integers $1, 2, \ldots, 2\rho+1$) is the perfect square of such a function. Write

$$\phi(x) \quad =(x-x_1)(x-x_2)\ldots(x-x_\rho) \quad \ldots \ldots \ldots \quad (8)$$

$$\left.\begin{aligned}-\mathrm{Q}(a_r) \quad &=l_r \quad (r=1, 2, \ldots, \rho)\\ \mathrm{P}(a_{\rho+s})&=l_{\rho+s}(s=1, 2, \ldots, \rho+1)\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (9)$$

then WEIERSTRASS defines

$$l_ral_r^2=\phi(a_r). \quad \ldots \ldots \ldots \quad (10)$$

for all values of r included in $1, 2, \ldots, 2\rho+1$. It is easy to verify that $x_1, x_2, \ldots, x_\rho$ are the roots of

$$\sum_{r=1}^{r=\rho}\left\{\frac{l_ral_r^2}{(a_r-x)\mathrm{P}'(a_r)}\right\}=1. \quad \ldots \ldots \ldots \quad (11)$$

for there are obviously ρ roots, and in order that x_1 may be one of these we must have

$$\sum_{r=1}^{r=\rho}\frac{(a_r-x_2)(a_r-x_3)\ldots(a_r-x_\rho)}{(a_r-a_1)(a_r-a_2)\ldots(a_r-a_\rho)}=1 \quad \ldots \ldots \ldots \quad (A).$$

By a known theorem of ABEL's we have

$$\Sigma\frac{z^s}{\frac{d\chi}{dz}}=0 \text{ or } 1,$$

according as $s <$ or $= \rho - 1$, the summation being extended over the ρ roots of $\chi(x) = 0$; and therefore

$$\sum_{r=1}^{r=\rho} \frac{a_r^{\rho-1}}{P'(a_r)} = 0$$

$$\sum_{r=1}^{r=\rho} \frac{a_r^{s}}{P'(a_r)} = 0 \text{ if } s < \rho - 1 \geqq 0.$$

Thus the left-hand side of (A)

$$= \sum_{r=1}^{r=\rho} \left\{ \frac{a_r^{\rho-1}}{P'(a_r)} - (x_2 + \ldots + x_\rho) \frac{a_r^{\rho-2}}{P'(a_r)} + \ldots \right\}$$

$$= 1$$

all the terms disappearing except the first and so verifying (A) and proving that $x_1, x_2, \ldots, x_\rho$ are the roots of (11).

15. Taking now our set of integrals u we have

$$\left.\begin{aligned}
0 &= \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \frac{P(x_\lambda)}{(x_\lambda - a_1)\sqrt{R(x_\lambda)}} \frac{\partial x_\lambda}{\partial u_\mu} \\
0 &= \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \frac{P(x_\lambda)}{(x_\lambda - a_2)\sqrt{R(x_\lambda)}} \frac{\partial x_\lambda}{\partial u_\mu} \\
&\cdots\cdots \\
1 &= \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \frac{P(x_\lambda)}{(x_\lambda - a_\mu)\sqrt{R(x_\lambda)}} \frac{\partial x_\lambda}{\partial u_\mu} \\
&\cdots\cdots \\
0 &= \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \frac{P(x_\lambda)}{(x_\lambda - a_\rho)\sqrt{R(x_\lambda)}} \frac{\partial x_\lambda}{\partial u_\mu}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (12).$$

Multiply these respectively by $\frac{l_1 a l_1^2}{P'(a_1)}, \frac{l_2 a l_2^2}{P'(a_2)}, \ldots$ and add; then, in virtue of equation (11),

$$\frac{l_\mu a l_\mu^2}{P'(a_\mu)} = -\tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \frac{P(x_\lambda)}{\sqrt{R(x_\lambda)}} \frac{\partial x_\lambda}{\partial u_\mu}$$

so that if we write

$$U = \tfrac{1}{2} \sum_{\lambda=1}^{\lambda=\rho} \int_{a_\lambda}^{x_\lambda} \frac{P(x)\,dx}{\sqrt{R(x)}} \quad \ldots\ldots\ldots \quad (13)$$

we have

$$\frac{l_\mu a l_\mu^2}{P'(a_\mu)} = -\frac{\partial U}{\partial u_\mu} \quad \ldots\ldots\ldots \quad (14).$$

U is obviously a symmetric function of the x's, and is therefore a function of the u's.

16. Again solving the equations (12) regarded as giving $\frac{\delta x_\lambda}{\delta u_\mu}$ for different values of λ, we have*

$$\tfrac{1}{2}\frac{\delta x_\lambda}{\delta u_\mu}=-\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{\phi(a_\mu)}{\mathrm{P}'(a_\mu)}\,\frac{1}{x_\lambda-a_\mu}$$

$$=-\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{l_\mu al_\mu^2}{(x_\lambda-a_\mu)\mathrm{P}'(a_\mu)}\quad\ldots\ldots\ldots\quad(15).$$

Therefore

$$\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\left[\frac{\phi'(x_\lambda)}{\sqrt{\mathrm{R}(x_\lambda)}}\,\frac{\delta x_\lambda}{\delta u_\mu}\right]=-\sum_{\mu=1}^{\mu=\rho}\frac{l_\mu al_\mu^2}{(x_\lambda-a_\mu)\mathrm{P}'(a_\mu)}$$

$$=1 \text{ by } (11)$$

and

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{1}{x_\lambda-a_s}\right\}=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\sum_{\mu=1}^{\mu=\rho}\frac{1}{x_\lambda-a_s}\,\frac{\delta x_\lambda}{\delta u_\mu}$$

$$=\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\frac{\delta}{\delta u_\mu}\left\{\log\prod_{\lambda=1}^{\lambda=\rho}(a_s-x_\lambda)\right\}$$

$$=\tfrac{1}{2}\sum_{\mu=1}^{\mu=\rho}\frac{\delta}{\delta u_\mu}\log\phi(a_s)$$

$$=\frac{1}{al_s}\sum_{\mu=1}^{\mu=\rho}\frac{\delta al_s}{\delta u_\mu}\quad\ldots\ldots\ldots\ldots\quad(16')$$

where s may be any of the integers 1, 2, . . . , $2\rho+1$.

Writing

$$\sum_{\mu=1}^{\mu=\rho}\frac{\delta al_s}{\delta u_\mu}=\overline{al}_s\quad\ldots\ldots\ldots\ldots\quad(17)$$

so that (16′) becomes

$$\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{1}{x_\lambda-a_s}\right\}=\frac{\overline{al}_s}{al_s}\quad\ldots\ldots\ldots\ldots\quad(16)$$

WEIERSTRASS defines

$$al_{r,s}=\frac{al_r\overline{al}_s-al_s\overline{al}_r}{a_r-a_s}$$

$$=\frac{al_r al_s}{a_r-a_s}\left[\frac{al_s}{\overline{al}_s}-\frac{\overline{al}_r}{al_r}\right]$$

$$=-\sum_{\lambda=1}^{\lambda=\rho}\left\{\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\,\frac{al_r al_s}{(x_\lambda-a_r)(x_\lambda-a_s)}\right\}\quad\ldots\ldots\ldots\quad(18)$$

where r, s must be different from each other, but otherwise may be any of the integers

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* *Cf.* SCOTT'S 'Determinants,' c. ix., §§ 11, 12.

$1, 2, \ldots, 2\rho+1$. Evidently $al_{r,s}=al_{s,r}$, and there are therefore $\rho(2\rho+1)$ functions $al_{r,s}$; these, together with the $2\rho+1$ functions al_s, are the functions of the theory. (They are, of course, not all independent; the complete system of relations among them may be found in the fifth section of the first chapter of the memoir already quoted.)

Further

$$\frac{1}{al_r}\frac{\delta al_r}{\delta u_s}=-\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\frac{1}{a_r-x_\lambda}\frac{\delta x_\lambda}{\delta u_s}$$

$$=\sum_{\lambda=1}^{\lambda=\rho}\frac{\sqrt{\mathrm{R}(x_\lambda)}}{\phi'(x_\lambda)}\frac{l_s al_s^2}{(x_\lambda-a_s)\mathrm{P}'(a_s)}\frac{1}{(a_r-x_\lambda)}$$

$$=\frac{l_s al_s^2}{\mathrm{P}'(a_s)}\frac{al_{r,s}}{al_r al_s}$$

and

$$\frac{\delta al_r}{\delta u_s}=\frac{l_s}{\mathrm{P}'(a_s)}al_s al_{r,s} \quad \ldots\ldots\ldots\ldots \quad (19)$$

in which

s may have any value $1, 2, \ldots, \rho$
r ,, ,, $1, 2, \ldots, 2\rho+1$,

but r, s may not be equal. If $r \leqq \rho$, this serves as a verification of (14).

Again, since $x_1, x_2, \ldots, x_\rho$ are the roots of (11),

$$\sum_{s=1}^{s=\rho}\frac{l_s al_s^2}{(x-a_s)\mathrm{P}'(a_s)}+1=\frac{(x-x_1)(x-x_2)\ldots(x-x_\rho)}{(x-a_1)(x-a_2)\ldots(x-a_\rho)}=\frac{\phi(x)}{\mathrm{P}(x)}.$$

In this write $x=a_{\rho+r}$, $(r=1, 2, \ldots, \rho+1)$; then

$$al^2_{\rho+r}=\frac{\phi(a_{\rho+r})}{l_{\rho+r}}$$

$$=1-\sum_{s=1}^{s=\rho}\frac{l_s al_s^2}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)} \quad \ldots\ldots\ldots \quad (20)$$

which expresses $\rho+1$ functions $al_{\rho+r}$ each in terms of the ρ functions $al_1, al_2, \ldots, al_\rho$. By (20) and (14) we have

$$al^2_{\rho+r}=1+\sum_{s=1}^{s=\rho}\frac{1}{a_s-a_{\rho+r}}\frac{\delta \mathrm{U}}{\delta u_s} \quad \ldots\ldots\ldots\ldots \quad (21).$$

[17. A simpler form can be given to this equation by the introduction of a series of $\rho+1$ new variables provisionally given by

$$\left.\begin{aligned}
u_1 &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_1-a_{\rho+r}} \\
u_2 &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_2-a_{\rho+r}} \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_\rho &= \sum_{r=1}^{r=\rho+1} \frac{u_{\rho+r}}{a_\rho-a_{\rho+r}}
\end{aligned}\right\}$$

These new u's are not fully determined: as the remaining equation necessary to determine them assume

$$f(u_{\rho+1}, \ldots, u_{2\rho+1})=0.$$

When substitution is made in U for $u_1, \ldots, u_\rho$, U will be a function of $u_{\rho+1}, \ldots, u_{2\rho+1}$; and we shall have

$$\sum_{r=1}^{r=\rho+1} \frac{\partial \mathrm{U}}{\partial u_{\rho+r}} \delta u_{\rho+r} = \sum_{s=1}^{s=\rho} \frac{\partial \mathrm{U}}{\partial u_s} \delta u_s$$

$$= \sum_{s=1}^{s=\rho} \sum_{r=1}^{r=\rho+1} \frac{1}{a_s-a_{\rho+r}} \frac{\partial \mathrm{U}}{\partial u_s} \delta u_{\rho+r}$$

and from the $(\rho+1)^{\text{th}}$ equation giving the new u's

$$\sum_{r=1}^{r=\rho+1} \frac{\partial f}{\partial u_{\rho+r}} \delta u_{\rho+r}=0.$$

Then by the principle of indeterminate multipliers

$$\frac{\partial \mathrm{U}}{\partial u_{\rho+r}} - \sum_{s=1}^{s=\rho} \frac{1}{a_s-a_{\rho+r}} \frac{\partial \mathrm{U}}{\partial u_s} = \lambda \frac{\partial f}{\partial u_{\rho+r}}$$

for all the $\rho+1$ values of r. Multiply these $\rho+1$ equations by $u_{\rho+1}, u_{\rho+2}, \ldots$ respectively and add; then

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial \mathrm{U}}{\partial u_{\rho+r}} - \sum_{s=1}^{s=\rho} u_s \frac{\partial \mathrm{U}}{\partial u_s} = \lambda \sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial f}{\partial u_{\rho+r}}.$$

Let the part of U which is of the order m in the u_s's $(s \leqq \rho)$ be denoted by U_m; then when expressed in terms of the $u_{\rho+r}$'s it still remains the term of order m, so that

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial \mathrm{U}_m}{\partial u_{\rho+r}} = m\mathrm{U}_m = \sum_{s=1}^{s=\rho} u_s \frac{\partial \mathrm{U}_m}{\partial u_s}$$

and summing up for the terms of all orders

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\partial \mathrm{U}}{\partial u_{\rho+r}} = \sum_{s=1}^{s=\rho} u_s \frac{\partial \mathrm{U}}{\partial u_s}$$

and therefore from the above

$$\lambda \sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta f}{\delta u_{\rho+r}} = 0,$$

equivalent to one of the two equations

$$\lambda = 0$$

or

$$\sum_{r=1}^{r=\rho+1} u_{\rho+r} \frac{\delta f}{\delta u_{\rho+r}} = 0.$$

The latter, taken with the equation

$$f(u_{\rho+1}, \ldots, u_{2\rho+1}) = 0,$$

implies that there is a homogeneous relation between the quantities $u_{\rho+r}$; this we may reject. The former leaves f arbitrary or non-existent, and so there would be only ρ equations to determine $\rho+1$ quantities, a difficulty, however, obviable at any time by assigning some new equation to make up the requisite number; but $\lambda=0$ simplifies the resulting equations in which it occurs, and therefore this is selected. Let us assume as our new equation

$$\frac{v}{b} = \frac{u_{\rho+1}}{b-a_{\rho+1}} + \frac{u_{\rho+2}}{b-a_{\rho+2}} + \ldots + \frac{u_{2\rho+1}}{b-a_{2\rho+1}}$$

where v is a quantity which may have a definite value assigned to it at any time, if desired. Thus we have

$$\frac{\delta U}{\delta u_{\rho+r}} = \sum_{s=1}^{s=\rho} \frac{1}{a_s - a_{\rho+r}} \frac{\delta U}{\delta u_s}$$

and therefore

$$a l^2_{\rho+r} = 1 + \frac{\delta U}{\delta u_{\rho+r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14')$$

similar in form to (14).

18. Let us obtain the new u's explicitly from the above equations. Writing

$$g(z) = (z-b)\mathrm{P}(z)$$

we have*

$$u_{\rho+r} = -\frac{g(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})} \left[\frac{\mathrm{Q}(a_1)}{g'(a_1)} \frac{u_1}{a_1 - a_{\rho+r}} + \ldots + \frac{\mathrm{Q}(a_\rho)}{g'(a_\rho)} \frac{u_\rho}{a_\rho - a_{\rho+r}} + \frac{\mathrm{Q}(b)}{g'(b)} \frac{\frac{v}{b}}{b - a_{\rho+r}} \right].$$

Now make b infinite, so that the assumed equation takes the form

$$u_{\rho+1} + u_{\rho+2} + \ldots + u_{2\rho+1} = v$$

and

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* *Cf.* SCOTT's '**Determinants**,' *l.c.*

$$\frac{g(a_{\rho+r})}{g'(a_s)}=\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{P}'(a_s)}$$

$$\frac{g(a_{\rho+r})}{b-a_{\rho+r}}=-\mathrm{P}(a_{\rho+r})$$

$$\frac{\mathrm{Q}(b)}{bg'(b)}=1$$

so that

$$-u_{\rho+r}=\sum_{s=1}^{s=\rho}\left\{\frac{u_s}{a_s-a_{\rho+r}}\,\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\mathrm{Q}(a_s)}{\mathrm{P}'(a_s)}\right\}-\frac{\mathrm{P}(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}v.$$

As a verification of these values we may deduce (14) from (14′) as follows :—

$$\sum_{s=1}^{s=\rho}\frac{\delta \mathrm{U}}{\delta u_s}\delta u_s=d\mathrm{U}=\sum_{r=1}^{r=\rho+1}\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}\delta u_{\rho+r}$$

$$=\sum_{r=1}^{r=\rho+1}\sum_{s=1}^{s=\rho}\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}\,\frac{l_s l_{\rho+r}}{\mathrm{P}'(a_s)\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta u_s}{a_s-a_{\rho+r}}+\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}\delta v.$$

Now the quantities δu_s, δv are independent ; hence we must have

$$\frac{\delta \mathrm{U}}{\delta u_s}=\sum_{r=1}^{r=\rho+1}\frac{l_s l_{\rho+r}}{\mathrm{P}'(a_s)\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\,\frac{\delta \mathrm{U}}{\delta u_{\rho+r}},$$

$$0=\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}.$$

Taking the second of these, we have

$$\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}=-1+al^2_{\rho+r}$$

$$=-1+\frac{\phi(a_{\rho+r})}{l_{\rho+r}}$$

and therefore

$$\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}=-\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}+\sum_{r=1}^{r=\rho+1}\frac{\phi(a_{\rho+r})}{\mathrm{Q}'(a_{\rho+r})}$$

$$=-1+1=0$$

by the theorem already quoted in the verification of equation (11). For the first summation we have

$$\sum_{r=1}^{r=\rho+1}\frac{l_s}{\mathrm{P}'(a_s)}\,\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\,\frac{\delta \mathrm{U}}{\delta u_{\rho+r}}$$

$$=\frac{l_s}{\mathrm{P}'(a_s)}\left[-\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}+\sum_{r=1}^{r=\rho+1}\frac{l_{\rho+r}al^2_{\rho+r}}{\mathrm{Q}'(a_{\rho+r})(a_s-a_{\rho+r})}\right].$$

The first term inside the bracket is the expansion in partial fractions of

$$-\frac{P(a_s)}{Q(a_s)}$$

and is therefore zero since $s \leqq \rho$; the second is

$$\sum_{r=1}^{r=\rho+1}\left\{\frac{\phi(a_{\rho+r})}{Q'(a_{\rho+r})}\,\frac{1}{a_s-a_{\rho+r}}\right\}$$

$$=\frac{\phi(a_s)}{Q(a_s)}=-al_s^2 \quad \text{by definition}$$

so that the equation with which we began leads to

$$\frac{\delta U}{\delta u_s}=-\frac{l_s}{P'(a_s)}al_s^2$$

that is, to equation (14).]

19. Now let

$$\left.\begin{aligned} V&=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=r}\int_{a_\lambda}^{\xi_\lambda}\frac{P(x)}{\sqrt{R(x)}}dx \\ W&=\tfrac{1}{2}\sum_{\lambda=1}^{\lambda=\rho}\int_{a_\lambda}^{p_\lambda}\frac{P(x)}{\sqrt{R(x)}}dx \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13')$$

so that V stands to the *v*'s and W to the *w*'s in exactly the same relation as U to the *u*'s.

Applying now the theorem in § 6 we have

$$U+V+W=-C_{\frac{1}{x}}\tfrac{1}{2}\frac{P(x)}{\sqrt{R(x)}}\log\left(\frac{My+Nz}{My-Nz}\right)$$

$$=-C_{\frac{1}{x}}\frac{y}{z}\left[\frac{Nz}{My}+\tfrac{1}{3}\left(\frac{Nz}{My}\right)^3+\ .\ .\ .\ \right].$$

The n^{th} term in this series gives

$$-\frac{1}{2n-1}C_{\frac{1}{x}}\left(\frac{N}{M}\right)^{2n-1}\left(\frac{z}{y}\right)^{2n-2}$$

$$=-\frac{1}{2n-1}C_{\frac{1}{x}}\left\{\frac{N_1^{2n-1}}{x^n}+\text{higher powers of }\frac{1}{x}\right\}$$

so that nothing is contributed except by the first term, and we have

$$U+V+W=-N_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

The 2ρ quantities $N_1, N_2, \ldots, N_\rho, M_1, \ldots, M_\rho$ are determined by the equation

$$N_1x^{\rho-1}z+N_2x^{\rho-2}z+\ .\ .\ .\ +N_\rho z+M_1x^{\rho-1}y+\ .\ .\ .\ +M_\rho y=-x^\rho y$$

which is satisfied by the 2ρ values of x, viz.: $x_1, x_2, \ldots, x_\rho, \xi_1, \xi_2, \ldots, \xi_\rho$; and therefore

$$\mathrm{N}_1 \begin{vmatrix} x_1^{\rho-1}z_1, x_1^{\rho-2}z_1, \ldots, y_1 \\ x_2^{\rho-1}z_2, x_2^{\rho-2}z_2, \ldots, y_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_\rho^{\rho-1}z_\rho, x_\rho^{\rho-2}z_\rho, \ldots, y_\rho \\ \xi_1^{\rho-1}\zeta_1, \xi_1^{\rho-2}\zeta_1, \ldots, \eta_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \xi_\rho^{\rho-1}\zeta_\rho, \xi_\rho^{\rho-2}\zeta_\rho, \ldots, \eta_\rho \end{vmatrix} + \begin{vmatrix} x_1^\rho y_1, x_1^{\rho-2}z_1, \ldots, y_1 \\ x_2^\rho y_2, x_2^{\rho-2}z_2, \ldots, y_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_\rho^\rho y_\rho, x_\rho^{\rho-2}z_\rho, \ldots, y_\rho \\ \xi_1^\rho \eta_1, \xi_1^{\rho-2}\zeta_1, \ldots, \eta_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \xi_\rho^\rho \eta_\rho, \xi_\rho^{\rho-2}\zeta_\rho, \ldots, \eta_\rho \end{vmatrix} = 0.$$

20. As an example of (22) and (23) consider the elliptic functions, *i.e.*, the case in which $\rho=1$; then

$$\mathrm{N}_1 \begin{vmatrix} z & y \\ \zeta & \eta \end{vmatrix} + \begin{vmatrix} xy & y \\ \xi\eta & \eta \end{vmatrix} = 0$$

(dropping suffixes), or

$$\mathrm{N}_1(z^2\eta^2 - y^2\zeta^2) = (\xi - x)y\eta(z\eta + y\zeta)$$

that is

$$\mathrm{N}_1\{(x-a_1)(\xi-a_1)-(a_1-a_2)(a_1-a_3)\} = -y\eta(z\eta+y\zeta);$$

and

$$u = \tfrac{1}{2}\int_{a_1}^{x} \frac{dx}{\sqrt{x-a_1 . x-a_2 . x-a_3}},$$

$$\mathrm{U} = \tfrac{1}{2}\int_{a_1}^{x} \frac{\sqrt{x-a_1}\,dx}{\sqrt{x-a_2\,x-a_3}}.$$

Let

$$x = a_1 + (a_2-a_1)t^2,\ k^2 = \frac{a_2-a_1}{a_3-a_1};$$

then

$$u = \frac{1}{\sqrt{a_3-a_1}}\int_0^t \frac{dt}{\sqrt{1-t^2\,1-k^2t^2}},$$

$$\mathrm{U} = \frac{a_2-a_1}{\sqrt{a_3-a_1}}\int_0^t \frac{t^2dt}{\sqrt{1-t^2 . 1-k^2t^2}}.$$

Let s, c, d denote elliptic functions of $u\sqrt{a_3-a_1}$
$\mathrm{S, C, D}$,, ,, $v\sqrt{a_3-a_1}$
σ ,, $\mathrm{sn}\{(u+v)\sqrt{a_3-a_1}\}$ *i.e.*, $-\mathrm{sn}w\sqrt{a_3-a_1}$.

Then

$$s^2 = t^2 = \frac{x-a_1}{a_2-a_1},$$

and therefore

$$\begin{aligned} x-a_1 &= (a_2-a_1)s^2 \\ x-a_2 &= -(a_2-a_1)c^2 \\ x-a_3 &= -(a_3-a_1)d^2 \end{aligned}$$

and so

$$\mathrm{N}_1=\frac{(a_2-a_1)^2\sqrt{a_3-a_1}\{scd\mathrm{S}^2+\mathrm{SCD}s^2\}}{(a_2-a_1)(a_3-a_1)(1-k^2s^2\mathrm{S}^2)}$$

$$=\frac{a_2-a_1}{\sqrt{a_3-a_1}}s\mathrm{S}\sigma.$$

With the ordinary notation for the second elliptic integral we have

$$\mathrm{E}(u\sqrt{a_3-a_1})=u\sqrt{a_3-a_1}-\frac{\mathrm{U}}{\sqrt{a_3-a_1}}$$

and since

$$u+v+w=0$$

this gives

$$-\frac{1}{\sqrt{a_3-a_1}}(\mathrm{U}+\mathrm{V}+\mathrm{W})=\mathrm{E}(u\sqrt{a_3-a_1})+\mathrm{E}(v\sqrt{a_3-a_1})+\mathrm{E}(w\sqrt{a_3-a_1}$$

$$=k^2s\mathrm{S}\sigma$$

$$=\frac{\mathrm{N}_1}{\sqrt{a_3-a_1}}$$

that is

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=-\mathrm{N}_1$$

agreeing with the case when $\rho=1$ of (22).

21. The evaluation of N_1 in terms of the functions can be obtained in the general case as follows.

Since $x_1, x_2, \ldots, x_\rho, \xi_1, \xi_2, \ldots, \xi_\rho, p_1, p_2, \ldots, p_\rho$ are the roots of

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2=0$$

we have

$$\mathrm{M}^2y^2-\mathrm{N}^2z^2\equiv(x-x_1)(x-x_2)\ldots(x-p_\rho).$$

In this write $x=a_m$, where $m\geqq1\leqq\rho$; then

$$-\mathrm{N}^2\mathrm{Q}(a_m)\equiv(a_m-x_1)(a_m-x_2)\ldots(a_m-p_\rho)$$

that is

$$l_m\{\mathrm{N}(a_m)\}^2=l_m{}^3al_m{}^2(u)al_m{}^2(v)al_m{}^2(u+v)$$

and therefore

$$\mathrm{N}_1a_m{}^{\rho-1}+\mathrm{N}_2a_m{}^{\rho-2}+\ldots+\mathrm{N}_\rho=\pm l_mal_m(u)al_m(v)al_m(u+v).$$

Hence

$$\sum_{m=1}^{m=\rho}\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)=\pm\sum_{m=1}^{m=\rho}\frac{\mathrm{N}_1a_m{}^{\rho-1}+\ldots+\mathrm{N}_\rho}{\mathrm{P}'(a_m)}.$$

But $a_1, a_2, \ldots a_\rho$ are the ρ roots of

$$\mathrm{P}(z)=0$$

and therefore as before

$$\sum_{m=1}^{m=\rho}\frac{a_m^{\rho-1}}{\mathrm{P}'(a_m)}=1 \text{ and } \sum_{m=1}^{m=\rho}\frac{a_m^{\rho-s}}{\mathrm{P}'(a_m)}=0(s>1\leqq\rho)$$

and so

$$\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}=\pm\mathrm{N}_1$$

and therefore

$$\begin{aligned}\mathrm{U}+\mathrm{V}+\mathrm{W}&=-\mathrm{N}_1\\&=\mp\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}.\end{aligned}$$

On the expansion of each side in terms of the u's and v's as is done below, it is at once seen that the lower sign is the correct one; and therefore

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=\sum_{m=1}^{m=\rho}\left\{\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\right\}.$$

This may be called the addition theorem for the integral-function; by putting $\rho=1$ and referring to the example worked out in the last section, it is at once seen to be the addition theorem for elliptic integrals of the second order.

22. In the expansion of the two sides in terms of u's and v's the first term is sufficient to indicate the correct sign in the above; but it is not uninteresting to see the agreement for terms of a higher order, and the expansion is carried on as far as the order seven in the magnitudes u.

Proceeding therefore to form the expansion of U in terms of the u's, write, with WEIERSTRASS,

$$\frac{\mathrm{P}'(a_r)}{l_r}(a_r-x_r)=s_r^2\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

so that

$$-\frac{\mathrm{P}'(a_r)}{l_r}dx_r=2s_rds_r$$

and

$$\frac{dx_r}{x_r-a_r}=2\frac{ds_r}{s_r}.$$

Let

$$\mathrm{a}_{r,m}=\frac{l_r}{\mathrm{P}'(a_r)}\,\frac{1}{a_r-a_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

so that

$$\frac{l_m}{\mathrm{P}'(a_m)}\mathrm{a}_{r,m}=-\frac{l_r}{\mathrm{P}'(a_r)}\mathrm{a}_{m,r};$$

then

$$x_r-a_m=(a_r-a_m)(1-\mathrm{a}_{r,m}s_r^2).$$

Substituting we have

$$\mathrm{P}(x_m)=-\frac{l_m s_m^2}{\mathrm{P}'(a_m)}\mathrm{P}'(a_m)+\left\{\frac{l_m s_m^2}{\mathrm{P}'(a_m)}\right\}^2\frac{\mathrm{P}''(a_m)}{2!}-\left\{\frac{l_m s_m^2}{\mathrm{P}'(a_m)}\right\}^3\frac{\mathrm{P}'''(a_m)}{3!}+\dots$$

$$\mathrm{Q}(x_m)=-l_m-\frac{l_m s_m^2}{\mathrm{P}'(a_m)}\mathrm{Q}'(a_m)+\left\{\frac{l_m s_m^2}{\mathrm{P}'(a_m)}\right\}^2\frac{\mathrm{Q}''(a_m)}{2!}-\dots$$

and therefore

$$\sqrt{\frac{\mathrm{P}(x_m)}{\mathrm{Q}(x_m)}}=s_m\sqrt{\frac{1-\dfrac{l_m s_m^2}{\{\mathrm{P}'(a_m)\}^2}\dfrac{\mathrm{P}''(a_m)}{2!}+\dots}{1+\dfrac{s_m^2}{\mathrm{P}'(a_m)}\mathrm{Q}'(a_m)-\dots}}$$

$$=s_m(1-\mathrm{A}_m s_m^2-\mathrm{B}_m s_m^4)\text{ say,} \quad \dots\dots \quad (26)$$

correct to the fifth order. Moreover

$$d\mathrm{U}=\tfrac{1}{2}\sum_{m=1}^{m=\rho}\sqrt{\frac{\mathrm{P}(x_m)}{\mathrm{Q}(x_m)}}dx_m$$

$$=-\sum_{m=1}^{m=\rho}\frac{l_m}{\mathrm{P}'(a_m)}s_m^2 ds_m(1-\mathrm{A}_m s_m^2-\mathrm{B}_m s_m^4)$$

or

$$\mathrm{U}=-\sum_{m=1}^{m=\rho}\frac{l_m}{\mathrm{P}'(a_m)}\left[\frac{s_m^3}{3}-\frac{\mathrm{A}_m}{5}s_m^5-\frac{\mathrm{B}_m}{7}s_m^7\right] \quad \dots\dots \quad (27)$$

correct to the seventh order. Further

$$2du_m=\sqrt{\frac{\mathrm{P}(x_1)}{\mathrm{Q}(x_1)}}\frac{dx_1}{x_1-a_m}+\dots+\sqrt{\frac{\mathrm{P}(x_m)}{\mathrm{Q}(x_m)}}\frac{dx_m}{x_m-a_m}+\dots$$

and

$$\frac{dx_r}{x_r-a_m}=-2\mathrm{a}_{r,m}s_r ds_r(1+\mathrm{a}_{r,m}s_r^2+\mathrm{a}_{r\,m}^2 s_r^4)$$

which with the help of (26) gives

$$du_m=ds_m(1-\mathrm{A}_m s_m^2-\mathrm{B}_m s_m^4)-\sum_{r=1}^{r=\rho}{}'\,\mathrm{a}_{r,m}s_r^2 ds_r(1+\mathrm{a}_{r,m}s_r^2+\mathrm{a}_{r,m}^2 s_r^4)(1-\mathrm{A}_r s_r^2-\mathrm{B}_r s_r^4)$$

where Σ' denotes that r may receive all values between 1 and ρ except m. Thus

$$u_m=s_m-\frac{\mathrm{A}_m}{3}s_m^3-\frac{\mathrm{B}_m}{5}s_m^5-\sum_{r=1}^{r=\rho}{}'\,\mathrm{a}_{r,m}\left\{\frac{s_r^3}{3}+\frac{\mathrm{a}_{r,m}-\mathrm{A}_r}{5}s_r^5\right\} \quad \dots\dots \quad (28)$$

correct to the fifth order; and as s_m is of the order u this expansion will be sufficient for the expansion of U in (27) accurately to the seventh order. The equation (28) holds for $m=1, 2, \dots, \rho$.

Inverting it in order to obtain s in terms of u's we find that to the fifth order

$$
\begin{aligned}
s_m = {} & u_m \\
& + \frac{A_m}{3} u_m^3 + \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} \frac{u_r^3}{3} \\
& + \frac{5A_m^2 + 3B_m}{15} u_m^5 + \frac{A_m}{3} u_m^2 \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} u_r^3 + \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} \frac{a_{r,m}(3a_{r,m} + 2A_r)}{15} u_r^5 + \tfrac{1}{3} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} u_r^2 \overset{s=\rho}{\underset{s=1}{\Sigma^{r}}} a_{s,r} u_s^3 \qquad (29)
\end{aligned}
$$

where $\overset{r=\rho}{\underset{r=1}{\Sigma^{m}}}$ implies summation for all values of r from 1 to ρ except m, and $\overset{s=\rho}{\underset{s=1}{\Sigma^{r}}}$ for all values of s from 1 to ρ except r. Substituting this in (27) we obtain

$$
\begin{aligned}
-U = \sum_{m=1}^{m=\rho} \frac{l_m}{P'(a_m)} \Bigg[& \frac{u_m^3}{3} + \frac{2A_m}{15} u_m^5 + u_m^2 \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} \frac{u_r^3}{3} + \left(\frac{2B_m}{35} + \frac{A_m^2}{9} \right) u_m^7 \\
& + \frac{2A_m}{3} u_m^4 \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} \frac{u_r^3}{3} + \frac{u_m^2}{15} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m}(3a_{r,m} + 2A_r) u_r^5 \\
& + \frac{u_m^2}{3} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} u_r^2 \overset{s=\rho}{\underset{s=1}{\Sigma^{r}}} a_{s,r} u_s^3 + \frac{u_m}{9} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} \overset{s=\rho}{\underset{s=1}{\Sigma^{m}}} a_{r,m} a_{s,m} u_r^3 u_s^3 \Bigg] \quad . \quad . \quad . \quad (30)
\end{aligned}
$$

correct to the seventh order. In the last term inside the bracket r and s may take the same value ; the double summation is in fact

$$
\left[\overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} u_r^3 \right]^2 .
$$

Again

$$
\begin{aligned}
al_m^2(u) & = \frac{1}{l_m} \phi(a_m) \\
& = s_m^2 (1 - a_{1,m} s_1^2)(1 - a_{2,m} s_2^2) \ . \ . \ . \ (1 - a_{\rho,m} s_\rho^2)
\end{aligned}
$$

(the term involving s_m^2 not occurring in the brackets)

$$
= s_m^2 \left[1 - \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} s_r^2 + \overset{r,t=\rho}{\underset{r,t=1}{\Sigma}} a_{r,m} a_{t,m} s_r^2 s_t^2 \right]
$$

where $\overset{r,t=\rho}{\underset{r,t=1}{\Sigma^{m}}}$ implies summation for all values 1, 2, . . . , ρ of r and t except m, and r and t must not have the same values. Extracting the square root we find

$$
al_m(u) = s_m \left[1 - \tfrac{1}{2} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m} s_r^2 - \tfrac{1}{8} \overset{r=\rho}{\underset{r=1}{\Sigma^{m}}} a_{r,m}^2 s_r^4 + \tfrac{1}{4} \overset{r,t=\rho}{\underset{r,t=1}{\Sigma^{m}}} a_{r,m} a_{t,m} s_r^2 s_t^2 \right] \quad . \quad . \quad . \quad (31).
$$

Let σ_m refer to $al_m(v)$, S_m to $al_m(u+v)$, so that to the first order

$$al_m(v)=\sigma_m$$
$$al_m(u+v)=\mathrm{S}_m$$

and regard σ_m and S_m as being of the same order as s_m. Then accurately to the seventh order

$$al_m(u)al_m(v)al_m(u+v)$$

$$=s_m\sigma_m\mathrm{S}_m\Big[1-\tfrac{1}{2}\sum_{r=1}^{r=\rho}{}^m\mathfrak{a}_{r,m}(s_r^2+\sigma_r^2+\mathrm{S}_r^2)-\tfrac{1}{8}\sum_{r=1}^{r=\rho}{}^m\mathfrak{a}_{r,m}^2(s_r^4+\sigma_r^4+\mathrm{S}_r^4)$$
$$+\tfrac{1}{4}\sum_{r,t=1}^{r,t=\rho}{}^m\mathfrak{a}_{r,m}\mathfrak{a}_{t,m}(s_r^2s_t^2+\sigma_r^2\sigma_t^2+\mathrm{S}_r^2\mathrm{S}_t^2)+\tfrac{1}{4}\sum_{r=1}^{r=\rho}{}^m\sum_{t=1}^{t=\rho}{}^m\mathfrak{a}_{r,m}\mathfrak{a}_{t,m}(s_r^2\sigma_t^2+\sigma_r^2\mathrm{S}_t^2+\mathrm{S}_r^2s_t^2)\Big] \quad (32)$$

where the summation in the last term in (32) is exactly as in the last term in (30). To express this in terms of u and v we must substitute the value of s in terms of u's as given by (29) and for σ and S respectively corresponding values of v and $u+v$. Let these values be inserted, both sides multiplied by $\frac{l_m}{\mathrm{P}'(a_m)}$ and the summation taken for the values $m=1$ to $m=\rho$ and compare this expression, which is

$$\sum_{m=1}^{m=\rho}\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_m(v)al_m(u+v)\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(\mathrm{B}),$$

with the value of U+V+W.

Firstly, they agree in the third order of quantities; for

$$-\frac{u_m^3+v_m^3+w_m^3}{3}=u_mv_m(u_m+v_m)$$

since

$$u_m+v_m+w_m=0.$$

Secondly, consider in each the terms of the order five. That in U+V+W which has $\frac{l_m}{\mathrm{P}'(a_m)}\mathrm{A}_m$ for its coefficient is

$$\tfrac{2}{15}[(u_m+v_m)^5-u_m^5-v_m^5]$$
$$=\tfrac{2}{3}u_mv_m(u_m^3+2u_m^2v_m+2u_mv_m^2+v_m^3)$$
$$=\tfrac{2}{3}u_mv_m(u_m+v_m)(u_m^2+v_m^2+u_mv_m)$$

while in (B) it is .

$$\tfrac{1}{3}u_mv_m(u_m+v_m)\{u_m^2+v_m^2+(u_m+v_m)^2\}$$

and these are obviously equal.

The term in U+V+W which has

$$\tfrac{1}{3}\frac{l_m}{\mathrm{P}'(a_m)}\frac{l_r}{\mathrm{P}'(a_r)}\frac{1}{a_r-a_m},\ i.e.\ \tfrac{1}{3}\frac{l_m}{\mathrm{P}'(a_m)}\mathfrak{a}_{r,m}\text{ or }-\tfrac{1}{3}\frac{l_r}{\mathrm{P}'(a_r)}\mathfrak{a}_{m,r},$$

for coefficient is

$$(u_m+v_m)^2(u_r+v_r)^3-(u_r+v_r)^2(u_m+v_m)^3-u_m^2u_r^3+u_r^2u_m^3-v_m^2v_r^3+v_r^2v_m^3$$

while in (B) it is

$$\begin{aligned}&(u_m+v_m)v_mu_r^3-(u_r+v_r)v_ru_m^3-\tfrac{3}{2}u_mv_m(u_m+v_m)\{u_r^2+v_r^2+(u_r+v_r)^2\}\\&+(u_m+v_m)u_mv_r^3-(u_r+v_r)u_rv_m^3+\tfrac{3}{2}u_rv_r(u_r+v_r)\{u_m^2+v_m^2+(u_m+v_m)^2\}\\&+u_mv_m(u_r+v_r)^3-u_rv_r(u_m+v_m)^3.\end{aligned}$$

Adding the latter up in columns, it is

$$\begin{aligned}&u_r^3[(u_m+v_m)^2-u_m^2]+v_r^3[(u_m+v_m)^2-v_m^2]+3u_mv_mu_rv_r(u_r+v_r)\text{ for first}\\&-u_m^3[u_r+v_r)^2-u_r^2]-v_m^3[(u_r+v_r)^2-v_r^2]-3u_mv_mu_rv_r(u_m+v_m)\text{ for second}\\&-3u_mv_m(u_m+v_m)(u_r+v_r)^2+3u_mv_mu_rv_r(u_m+v_m)+3u_rv_r(u_r+v_r)(u_m+v_m)^2\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-3u_rv_ru_mv_m(u_r+v_r)\text{ for third}\\&=-u_m^2u_r^3-v_m^2v_r^3+u_m^3u_r^2+v_m^3v_r^2+(u_m+v_m)^2(u_r+v_r)^3-(u_r+v_r)^2(u_m+v_m)^3\end{aligned}$$

and therefore, to the order five, (B) and U+V+W are equal.

Thirdly, consider in the order seven the term in U+V+W which has

$$\frac{l_m}{\mathrm{P}'(a_m)}\left(\frac{2\mathrm{B}_m}{35}+\frac{\mathrm{A}_m^2}{9}\right)$$

for coefficient; it is

$$\begin{aligned}&(u_m+v_m)^7-u_m^7-v_m^7\\&=7u_mv_m(u_m+v_m)[u_m^4+v_m^4+3u_m^2v_m^2+2u_mv_m(u_m^2+v_m^2)]\end{aligned}$$

while in B the term of order seven which is free from all the a's and is multiplied by $\frac{l_m}{\mathrm{P}'(a_m)}$ is

$$\begin{aligned}&\frac{l_m}{\mathrm{P}'(a_m)}[u_mv_m(u_m+v_m)\{u_m^4+v_m^4+(u_m+v_m)^4\}]\frac{5\mathrm{A}_m^2+3\mathrm{B}_m}{15}\\&+\frac{l_m}{\mathrm{P}'(a_m)}\,\frac{\mathrm{A}_m^2}{9}[u_mv_m(u_m+v_m)\{v_m^2(u_m+v_m)^2+u_m^2(u_m+v_m)^2+u_m^2v_m^2\}]\\&=\frac{l_m}{\mathrm{P}'(a_m)}\left(\frac{2\mathrm{B}_m}{5}+\frac{7\mathrm{A}_m^2}{9}\right)u_mv_m(u_m+v_m)[u_m^4+v_m^4+3u_m^2v_m^2+2u_mv_m(u_m^2+v_m^2)]\end{aligned}$$

and again these terms are equal.

I have verified the exact agreement of the two expressions for one or two others (but not for all, owing to the labour involved) of the terms of the seventh order; and this exact agreement leads us to infer the truth of the equation

$$U+V+W=\sum_{m=1}^{m=\rho}\frac{l_m}{P'(a_m)}al_m(u)al_m(v)al_m(u+v)\quad .\ .\ .\ .\ .\quad (33)$$

a direct proof of which has already been obtained.

23. Combining (22) and (33) we have

$$U+V+W=-N_1=\sum_{s=1}^{s=\rho}\frac{l_s}{P'(a_s)}al_s(u)al_s(v)al_s(u+v).$$

Therefore

$$\sum_{m=1}^{m=\rho}\left[\frac{\delta U}{\delta u_m}\delta u_m+\frac{\delta V}{\delta v_m}\delta v_m+\frac{\delta W}{\delta w_m}\delta w_m\right]=-\sum_{m=1}^{m=\rho}\left[\frac{\delta N_1}{\delta u_m}\delta u_m+\frac{\delta N_1}{\delta v_m}\delta v_m\right].$$

But by (7)

$$\delta u_m+\delta v_m+\delta w_m=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (7')$$

so that substituting for the δw's and remembering that the δu's and δv's are independent we have

$$\left.\begin{aligned}\frac{\delta U}{\delta u_m}-\frac{\delta W}{\delta w_m}&=-\frac{\delta N_1}{\delta u_m}\\ \frac{\delta V}{\delta v_m}-\frac{\delta W}{\delta w_m}&=-\frac{\delta N_1}{\delta v_m}\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (34).$$

By the first of these

$$\frac{\delta^2 U}{\delta u_m\delta u_n}\delta u_n-\frac{\delta^2 W}{\delta w_m\delta w_n}\delta w_n=-\frac{\delta^2 N_1}{\delta u_m\delta u_n}\delta u_n-\frac{\delta^2 N_1}{\delta u_m\delta v_n}\delta v_n$$

and therefore by (7′)

$$\begin{aligned}\frac{\delta^2 U}{\delta u_m\delta u_n}+\frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta u_m\delta u_n}\\ \frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta u_m\delta v_n}\end{aligned}$$

Similarly

$$\left.\begin{aligned}\frac{\delta^2 V}{\delta v_m\delta v_n}+\frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta v_m\delta v_n}\\ \frac{\delta^2 W}{\delta w_m\delta w_n}&=-\frac{\delta^2 N_1}{\delta v_m\delta u_n}\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (35)$$

from which we see that N_1 satisfies the series of differential equations

$$\frac{\delta^2 N_1}{\delta u_m\delta v_n}=\frac{\delta^2 N_1}{\delta v_m\delta u_n}$$

of which there are $\frac{1}{2}\rho(\rho-1)$ in all.

24. Returning now to (34) and using (14) we have

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u+v)-al_m^2(u)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(v)\left\{al_s(u)\frac{\delta al_s(u+v)}{\delta u_m}+al_s(u+v)\frac{\delta al_s u}{\delta u_m}\right\} \quad . \ (36)$$

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u+v)-al_m^2(v)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u)\left\{al_s(v)\frac{\delta al_s(u+v)}{\delta v_m}+al_s(u+v)\frac{\delta al_s(v)}{\delta v_m}\right\} \ . \ (37)$$

and from these by subtraction and noticing that

$$\frac{\delta al_s(u+v)}{\delta u_m}=\frac{\delta al_s(u+v)}{\delta v_m}$$

we have

$$\frac{l_m}{\mathrm{P}'(a_m)}\{al_m^2(u)-al_m^2(v)\}=\sum_{s=1}^{s=\rho}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u+v)\left\{al_s(u)\frac{\delta al_s(v)}{\delta v_m}-al_s(v)\frac{\delta al_s(u)}{\delta u_m}\right\} \quad . \quad (38).$$

Now if s be different from m

$$\frac{\delta al_s(u)}{\delta u_m}=\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)al_{s,m}(u)$$

but this no longer holds when s,m are the same since $al_{m,m}$ is not a recognised function. We proceed as follows to obtain $\frac{\delta al_m(u)}{\delta u_m}$:—differentiate both sides of (20) with respect to u_m so that

$$al_{\rho+r}(u)\frac{\delta al_{\rho+r}(u)}{\delta u_m}=-\overset{s=\rho}{\underset{s=1}{\Sigma^m}}\left\{\frac{l_s}{a_s-a_{\rho+r}}\frac{al_s(u)}{\mathrm{P}'(a_s)}\frac{\delta al_s(u)}{\delta u_m}\right\}-\frac{l_m}{a_m-a_{\rho+r}}\frac{al_m(u)}{\mathrm{P}'(a_m)}\frac{\delta al_m(u)}{\delta u_m}$$

where $\overset{s=\rho}{\underset{s=1}{\Sigma^m}}$ implies that the value $s=m$ is not to be included in the summation. The equation quoted above (holding for all values of s from 1 to $2\rho+1$) when substituted in the last gives, on division by $\frac{l_m}{\mathrm{P}'(a_m)}al_m(u)$,

$$al_{m,\rho+r}(u)al_{\rho+r}(u)+\overset{s=\rho}{\underset{s=1}{\Sigma^m}}\left\{\frac{l_s}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s(u)al_{s,m}(u)\right\}=-\frac{1}{a_m-a_{\rho+r}}\frac{\delta al_m(u)}{\delta u_m}$$

and (38) may now be written in the form

$$\begin{aligned}al_m^2(u)-al_m^2(v)=&\overset{s=\rho}{\underset{s=1}{\Sigma^m}}\frac{l_s}{\mathrm{P}'(a_s)}al_s(u+v)\{al_s(u)al_m(v)al_{s,m}(v)-al_s(v)al_m(u)al_{s,m}(u)\}\\ &-al_m(u+v)\overset{s=\rho}{\underset{s=1}{\Sigma^m}}\frac{l_s(a_m-a_{\rho+r})}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s(v)al_{s,m}(v)al_m(u)-al_s(u)al_{s,m}(u)al_m(v)\\ &-(a_m-a_{\rho+r})al_m(u+v)\{al_m(u)al_{\rho+r}(v)al_{m,\rho+r}(v)-al_m(v)al_{\rho+r}(u)al_{m,\rho+r}(u)\} \quad (39).\end{aligned}$$

This equation holds for the values 1, 2, . . . , ρ of m and these ρ equations determine the ρ functions $al_m(u+v)$ for values 1, . . . , ρ of m in terms of functions of u and v. Moreover, r is any one of the numbers 1, 2, . . . , $\rho+1$, so that these equations can have a large variety of forms. We may thus consider the functions $al_m(u+v)(m\leqq\rho)$ as known; the $\rho+1$ functions $al_{\rho+r}(u+v)$ are given in terms of them and therefore ultimately in terms of functions of u and v by the equation

$$al^2_{\rho+r}(u+v)=1-\sum_{s=1}^{s=\rho}{}'_{m}\left\{\frac{l_r}{(a_s-a_{\rho+r})\mathrm{P}'(a_s)}al_s^2(u+v)\right\}.$$

Treating (36) in the same manner as (38) it will yield ρ equations involving the double-suffix functions of $u+v$; this system, together with the relations between them (to which reference has already been made), will furnish the complete solution of the addition theorem for these functions.

Abelian functions of order 2.

25. Consider the particular case of the preceding for which $\rho=2$. We now have

$$\left.\begin{aligned}\mathrm{P}(x)&=(x-a_1)(x-a_2)\\ \mathrm{Q}(x)&=(x-a_3)(x-a_4)(x-a_5)\\ \mathrm{R}(x)&=\mathrm{P}(x)\mathrm{Q}(x)\end{aligned}\right\}$$

$$\left.\begin{aligned}u_1&=\tfrac{1}{2}\int_{a_1}^{x_1}\frac{x-a_2}{\sqrt{\mathrm{R}(x)}}\,dx+\tfrac{1}{2}\int_{a_2}^{x_2}\frac{x-a_2}{\sqrt{\mathrm{R}(x)}}\,dx\\ u_2&=\tfrac{1}{2}\int_{a_1}^{x_1}\frac{x-a_1}{\sqrt{\mathrm{R}(x)}}\,dx+\tfrac{1}{2}\int_{a_2}^{x_2}\frac{x-a_1}{\sqrt{\mathrm{R}(x)}}\,dx\end{aligned}\right\}$$

$$\phi(x)=(x-x_1)(x-x_2).$$

Write

$$\left.\begin{aligned}x_1-a_r&=\mathrm{a}_r\\ x_2-a_r&=\mathrm{b}_r\end{aligned}\right\}(r=1,\ 2,\ 3,\ 4,\ 5)\quad .\ .\ .\ .\ .\ .\ .\ .\quad (1).$$

Also

$$\left.\begin{aligned}l_1,\ l_2&=-\mathrm{Q}(a_1),\ -\mathrm{Q}(a_2)\\ l_3,\ l_4,\ l_5&=\mathrm{P}(a_3),\ \mathrm{P}(a_4),\ \mathrm{P}(a_5)\end{aligned}\right\}\text{respectively}\quad .\ .\ .\ .\ .\ .\quad (2).$$

Then

$$l_s al_s^2=\phi(a_s)=\mathrm{a}_s\mathrm{b}_s\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (3)$$

for $s=1, 2, 3, 4, 5$; and

$$al_{r,s}=\frac{1}{(r_1-x_2)\sqrt{l_rl_s}}\left[\sqrt{\mathrm{a}_r\mathrm{a}_s\mathrm{bbb}}-\sqrt{\mathrm{b}_r\mathrm{b}_s\mathrm{aaa}}\right]\quad .\ .\ .\ .\ .\ .\quad (4)$$

the suffixes being added to the a's and b's under the radical sign so as to have 1, 2, 3, 4, 5 for the complete system under any one root-sign. Then

$$al_t al_{r,s} = \frac{1}{(x_1-x_2)\sqrt{l_r l_s l_t}}\left[\mathrm{b}_t\sqrt{\mathrm{a}_r\mathrm{a}_s\mathrm{a}_t\mathrm{bb}} - \mathrm{a}_t\sqrt{\mathrm{b}_r\mathrm{b}_s\mathrm{b}_t\mathrm{aa}}\right]$$

and therefore

$$(a_r-a_s)al_t al_{r,s} + (a_s-a_t)al_r al_{s,t} + (a_t-a_r)al_s al_{r,t} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again

$$l_s al_s al_{r,s} = \frac{1}{(x_1-x_2)\sqrt{l_s}}\left[\mathrm{a}_r\sqrt{\mathrm{a}_s\mathrm{bbbb}} - \mathrm{b}_r\sqrt{\mathrm{b}_s\mathrm{aaaa}}\right]$$

and therefore

$$(a_p-a_q)l_r al_r al_{r,s} + (a_q-a_r)l_p al_p al_{p,s} + (a_r-a_p)l_q al_q al_{q,s} = 0 \quad . \quad . \quad . \quad . \quad (6)$$

in which p, q, r, s, t may be any of the numbers 1, 2, 3, 4, 5.

26. Writing $\frac{l_s}{\mathrm{P}'(a_s)} = \alpha_s$ $(s=1,\ 2)$, equation (39) of the last example gives

$$\begin{aligned} al_1^2(u) - al_1^2(v) = {} & \alpha_2 al_2(u+v)\{al_1(v)al_2(u)al_{1,2}(v) - al_1(u)al_2(v)al_{1,2}(u)\} \\ & + al_1(u+v)\Big[(a_1-a_3)\{al_1(v)al_3(u)al_{1,3}u - al_1(u)al_3(v)al_{1,3}(v)\} \\ & \qquad + \alpha_2\frac{a_1-a_3}{a_2-a_3}\{al_1(v)al_2(u)al_{1,2}(u) - al_1(u)al_2(v)al_{1,2}(v)\}\Big] \end{aligned}$$

$$\begin{aligned} al_2^2(u) - al_2^2(v) = {} & \alpha_1 al_1(u+v)\{al_1(u)al_2(v)al_{1,2}(v) - al_1(v)al_2(u)al_{1,2}(u)\} \\ & + al_2(u+v)\Big[(a_2-a_3)\{al_2(v)al_3(u)al_{2,3}(u) - al_2(u)al_3(v)al_{2,3}(v)\} \\ & \qquad + \alpha_1\frac{a_2-a_3}{a_1-a_3}\{al_1(u)al_2(v)al_{1,2}(u) - al_1(v)al_2(u)al_{1,2}(v)\}\Big] \end{aligned}$$

two equations which determine $al_1(u+v)$, $al_2(u+v)$.

Assuming these known we have

$$al_3^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_3}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_3}al_2^2(u+v)\right\},$$

$$al_4^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_4}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_4}al_2^2(u+v)\right\},$$

$$al_5^2(u+v) = 1 - \left\{\frac{\alpha_1}{a_1-a_5}al_1^2(u+v) + \frac{\alpha_2}{a_2-a_5}al_2^2(u+v)\right\}.$$

The equation (36) applied to this case is when $m=1$

$$\begin{aligned}al_1^2(u+v)-al_1^2(u)&=\alpha_2 al_2(v)\{al_1(u+v)al_2(u)al_{1,2}(u+v)+al_2(u+v)al_1(u)al_{1,2}(u)\}\\&\quad-(a_1-a_3)al_1(v)\{al_1(u)al_3(u+v)al_{1,3}(u+v)+al_1(u+v)al_3(u)al_{1,3}(u)\}\\&\quad-\alpha_2\frac{a_1-a_3}{a_2-a_3}al_1(v)\{al_1(u)al_2(u+v)al_{1,2}(u+v)+al_1(u+v)al_2(u)al_{1,2}(u)\}\end{aligned}$$

and when $m=2$ it is

$$\begin{aligned}al_2^2(u+v)-al_2^2(u)&=\alpha_1 al_1(v)\{al_1(u)al_2(u+v)al_{1,2}(u+v)+al_1(u+v)al_2(u)al_{1,2}(u)\}\\&\quad-(a_2-a_3)al_2(v)\{al_2(u)al_3(u+v)al_{2,3}(u+v)+al_2(u+v)al_3(u)al_{2,3}(u)\}\\&\quad-\alpha_1\frac{a_2-a_3}{a_1-a_3}al_2(v)\{al_2(u)al_1(u+v)al_{1,2}(u+v)+al_2(u+v)al_1(u)al_{1,2}(u)\}.\end{aligned}$$

A particular case of (5) is

$$(a_1-a_2)al_3al_{1,2}+(a_2-a_3)al_1al_{2,3}+(a_3-a_1)al_2al_{1,3}=0.$$

These three equations will suffice to determine $al_{1,2}(u+v)$, $al_{2,3}(u+v)$, $al_{3,1}(u+v)$; after which the other functions may be successively obtained from the equations

$$\begin{aligned}(a_2-a_3)l_4al_4al_{1,4}&=(a_2-a_4)l_3al_3al_{1,3}+(a_4-a_3)l_2al_2al_{1,2} \quad . \quad . \quad . \quad . \quad (6')\\(a_4-a_2)al_1al_{2,4}&=(a_4-a_1)al_2al_{1,4}+(a_1-a_2)al_4al_{1,2} \quad . \quad . \quad . \quad . \quad . \quad (5')\\(a_4-a_3)al_2al_{3,4}&=(a_4-a_2)al_3al_{2,4}+(a_2-a_3)al_4al_{2,3} \quad . \quad . \quad . \quad . \quad . \quad (5')\\(a_2-a_4)l_5al_5al_{1,5}&=(a_2-a_5)l_4al_4al_{1,4}+(a_5-a_4)l_2al_2al_{1,2} \quad . \quad . \quad . \quad . \quad (6')\\(a_4-a_5)al_1al_{4,5}&=(a_1-a_5)al_4al_{1,5}+(a_4-a_1)al_5al_{1,4} \quad . \quad . \quad . \quad . \quad . \quad (5')\\(a_3-a_5)al_4al_{3,5}&=(a_3-a_4)al_5al_{3,4}+(a_4-a_5)al_3al_{4,5} \quad . \quad . \quad . \quad . \quad . \quad (5')\\(a_2-a_5)al_3al_{2,5}&=(a_2-a_3)al_5al_{2,3}+(a_3-a_5)al_2al_{3,5} \quad . \quad . \quad . \quad . \quad . \quad (5')\end{aligned}$$

the figure at the end of each line denoting from which of the equations (5) and (6) the particular line has been derived.

This case has been added and all the necessary equations have been written down as a justification of the statement made at the end of § 24.

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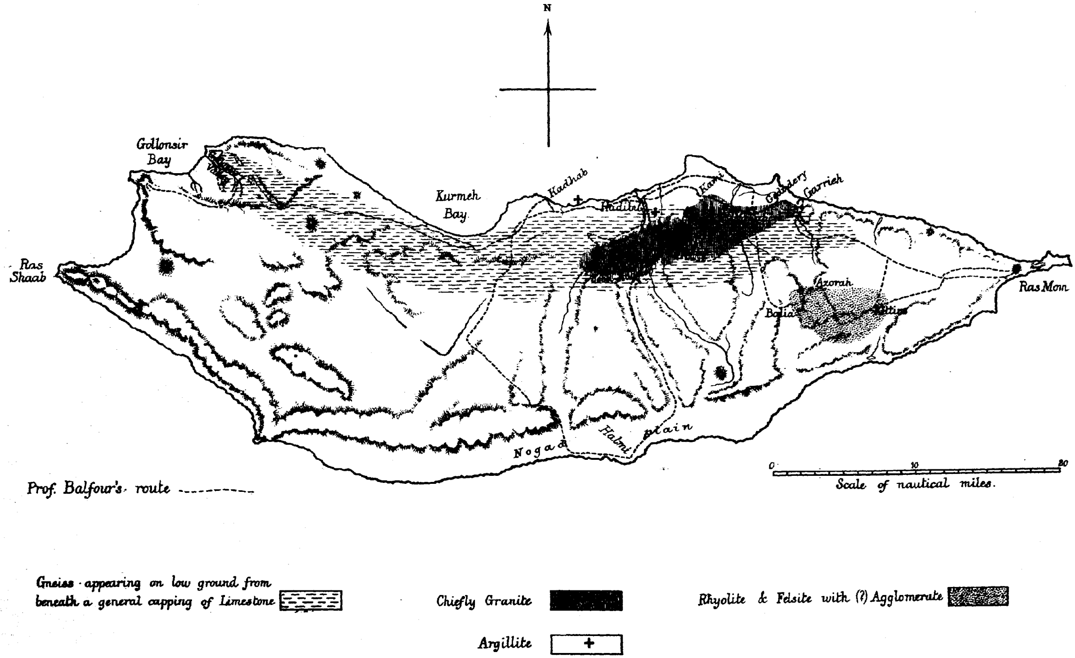
W. H. Wesley lith.

SOCOTRA.



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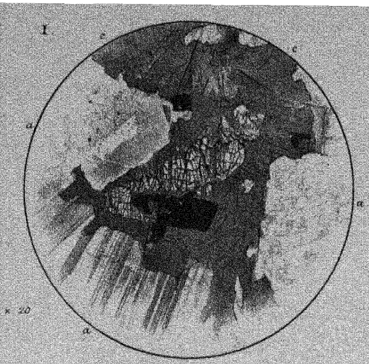
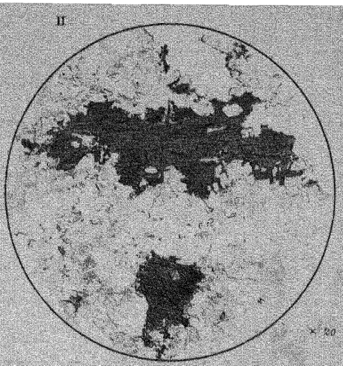


II



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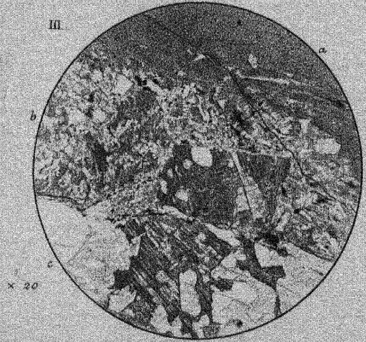

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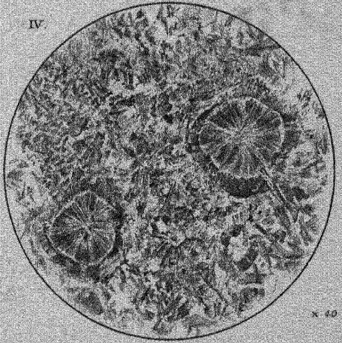

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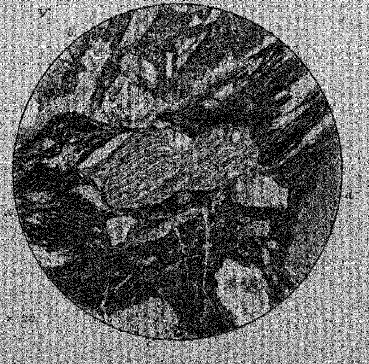

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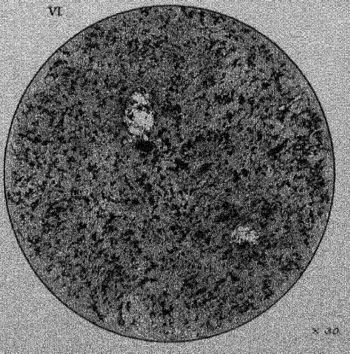

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